NUMERICAL VERIFICATION OF BEILINSON'S CONJECTURE FOR K_2 OF HYPERELLIPTIC CURVES.

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Abstract. We construct families of hyperelliptic curves over $\mathbb{Q}$ of arbitrary genus g with (at least) g integral elements in K_2. We also verify the Beilinson conjectures about K_2 numerically for several curves with $g = 2$, 3, 4 and 5. The first few sections of the paper also provide an elementary introduction to the Beilinson conjectures for K_2 of curves.

1. Introduction

Let k be a number field, with r_1 real embeddings and $2r_2$ complex embeddings into $\mathbb{C}$, so that $[k:\mathbb{Q}] = r_1 + 2r_2$. It is a well known classical theorem that, if $\mathcal{O}_k$ is the ring of algebraic integers in k, then $\mathcal{O}_k^*$ is a finitely generated abelian group of rank $r = r_1 + r_2 - 1$. If $u_1, \ldots, u_r$ form a basis of $\mathcal{O}_k^*$/torsion, and $\sigma_1, \ldots, \sigma_{r+1}$ are the complex embeddings of k up to complex conjugation, then the *regulator* of $\mathcal{O}_k^*$ is defined by

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right| ,$$

and one has that $\mathrm{Res}_{s=1}\zeta_k(s) = \dfrac{2^{r_1}(2\pi)^{r_2} R\,|\mathrm{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$, where Δ_k is the absolute value of the discriminant of k and w the number of roots of unity in k. Because $K_0(\mathcal{O}_k) \cong \mathrm{Cl}(\mathcal{O}_k) \oplus \mathbb{Z}$ and $K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$, so $|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$ and $w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$, this can be interpreted as a statement about the K-theory of $\mathcal{O}_k$, and it is from this point of view that it can be generalized to $\zeta_k(n)$ for $n \geq 2$. Namely, in [14], Quillen proved that $K_n(\mathcal{O}_k)$ is a finitely generated abelian group for all n. Borel in [4] computed its rank, showing that this rank is zero for even $n \geq 2$ and is equal to $r_\pm$ for odd $n = 2m-1 > 1$, where $(-1)^m = \pm 1$ and $r_- = r_1 + r_2$, $r_+ = r_2$. Moreover, for those odd n he showed (see [5]) that a suitably defined regulator of $K_{2m-1}(\mathcal{O}_k)$ is a non-zero rational multiple of $\zeta_k(m)/\pi^{mr_\mp}\sqrt{\Delta_k}$.

Inspired by this, Bloch in [2] considered K_2 of elliptic curves E defined over $\mathbb{Q}$ with complex multiplication, and showed that there is a relation between a regulator associated to $K_2(E)$ and the value of $L(E,2)$. Beilinson then proposed a very general conjecture about similar relations between certain regulators of K-groups of projective varieties over number fields and values of their L-functions at integers (see [15, §5]).

Those conjectures were tested numerically for K_2 of elliptic curves over $\mathbb{Q}$ by Bloch and Grayson in [3], which led to a modification of Beilinson's original conjecture, as explained in Section 3. Some further numerical work has been done in this

direction, e.g., Young carried out similar calculations for elliptic curves over certain real quadratic number fields (as well as over $\mathbb{Q}$) in his thesis [17], and Kimura worked out the case of the genus two curve $y^2 - y = x^5$ in [11].

The goal of this paper is to verify Beilinson's conjecture numerically for K_2 of a number of hyperelliptic curves over $\mathbb{Q}$ of genus greater than 1. The results of the computations support the predictions of Beilinson's conjecture in the cases we studied. Specifically, the conjecture says that the rank of a certain torsion-free abelian group $K_2(C;\mathbb{Z})$ associated to the curve C over $\mathbb{Q}$ is equal to the genus g of C, and that the associated regulator (the determinant of a certain $g \times g$-matrix) is rationally proportional to the appropriately normalized value of $L(C, 2)$ (see Section 3). If $g > 1$ it is quite difficult to write down enough elements in $K_2(C;\mathbb{Z})$, but we give several infinite families of hyperelliptic curves of genus 2 and 3 and one further infinite family for every genus $g \geq 2$, as well as sporadic examples for $g = 4$ or 5, for which we can construct at least g elements of $K_2(C;\mathbb{Z})$. For some 200 of those curves we check by computer that the regulator of g of those elements is non-zero and is related to the L-function of the curve in the expected way. Since the verification that a real number is non-zero can be done numerically, these calculations prove rigorously that the g elements in question are linearly independent and hence that $\operatorname{rk} K_2(C;\mathbb{Z}) \geq g$ for these curves. The relationship between the regulator and $L(C, 2)$, on the other hand, can only be established numerically to high precision.

Furthermore, the second author has shown, in a sequel to this paper ([7]), that in the family for arbitrary $g \geq 2$ there are, for each g, infinitely many non-isomorphic curves for which the regulator of the g elements is non-zero, thus showing that $\operatorname{rk} K_2(C;\mathbb{Z}) \geq g$ for those curves. For a more precise formulation of those (and other) results we refer the reader to Remark 10.11 or [7].

Our results also provide evidence for the reverse inequality $\operatorname{rk} K_2(C;\mathbb{Z}) \leq g$ predicted by Beilinson's conjecture. Namely, in a number of sporadic examples, and one universal situation (see Remark 10.13 for the precise list), we actually construct more than g elements of $K_2(C;\mathbb{Z})$. In each case the computer calculations of the regulator suggested a linear dependence over $\mathbb{Z}$ for any $g + 1$ of our elements. In most cases, including the universal case, these relations could then be proved; in the few remaining cases we have only a high-precision numerical verification at the regulator level. Since there is no intrinsic reason why the elements we construct should be linearly dependent when there are more than g of them, these results can be seen as evidence for the prediction that the rank of $K_2(C;\mathbb{Z})$ is at most g.

The structure of the paper is as follows. In Sections 2 and 3 we review the statement of Beilinson's conjecture for the case of K_2 of curves defined over $\mathbb{Q}$. In the next four sections we show how to construct curves with interesting elements in K_2. In Section 8 we deal with a technical condition in Beilinson's conjecture, the *integrality condition*, and in Section 9 we discuss how to compute the regulator numerically. The final section is devoted to examples.

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2. Curves and their L-functions (review)

Let C be a non-singular, projective, geometrically irreducible curve over $\mathbb{Q}$ of genus g. Its L-function is the Dirichlet series defined for $\mathrm{Re}(s) > \frac{3}{2}$ by the Euler product

$$L(C,s) = \prod_{p \text{ prime}} L_p(C,s) ,$$

where the Euler factor $L_p(C,s)$, for those primes p for which the equations defining C over $\mathbb{Q}$ can be reduced modulo p and still define a smooth curve of genus g over $\mathbb{F}_p$, is defined by the formula

$$L_p(C,s) = \exp\left(\sum_{n=1}^{\infty} \left(p^n + 1 - \#C(\mathbb{F}_{p^n})\right) \frac{p^{-ns}}{n} \right)$$

and is known to be the reciprocal of a polynomial in p^{-s} of degree $2g$ with constant term 1. For the remaining (finitely many) primes p, $L_p(C,s)^{-1}$ is also a polynomial in p^{-s} with constant term 1, but now of degree at most $2g$. Finally, one can associate to the curve $C/\mathbb{Q}$ a positive integer N called its *conductor* which plays a role in the expected functional equation of its L-function (see below). We omit the precise definitions of the remaining Euler factors and of N since they are a little complicated and since these quantities can be (and in some cases actually were) computed experimentally by assuming the functional equation rather than from their definitions. This functional equation is:

Conjecture 2.1. (Hasse-Weil) *The function*

$$L^*(C,s) = \frac{N^{s/2}}{(2\pi)^{gs}} \Gamma(s)^g L(C,s)$$

extends to an entire function of s and satisfies $L^*(C,s) = wL^*(C,2-s)$, where $w = +1$ or -1.

Note that, if this conjecture holds, then $L^{(0)}(C,0) = \cdots = L^{(g-1)}(C,0) = 0$ and

$$\frac{L^{(g)}(C,0)}{g!} = \lim_{s\to 0} \frac{L(C,s)}{s^g} = L^*(C,0) = wL^*(C,2) = \frac{wN}{(2\pi)^{2g}} L(C,2) \neq 0 ,$$

because $\Gamma(s)$ has a pole of order one at $s = 0$ with residue 1.

Remark 2.2. In this paper we need the value of $L(C,s)$ only at $s = 2$, where the defining Euler product is absolutely convergent. However, this convergence is very slow and in practice we will calculate the value of $L(C,2)$ numerically later on by assuming that Conjecture 2.1 holds and using the algorithms described in [9] and [8]. These algorithms also compute N, w, and the Euler factors $L_p(C,s)$ for "bad" primes under the assumption that the functional equation of $L^*(C,s)$ holds for some choices of these quantities from a finite list of possibilities, and at the same time check the validity of the resulting functional equation for $L^*(C,s)$ to high precision.

3. K-theory, regulators, and the Beilinson conjectures

In this section we will give definitions of the various K-groups occurring that are more elementary and explicit than the usual ones, but that are equivalent in our situation. We will indicate the relations to the standard definitions as we go along.

Let F be a field. Then $K_2(F)$ can be defined[1] as

$$F^* \otimes_{\mathbb{Z}} F^* / \langle a \otimes (1-a), a \in F, a \neq 0, 1 \rangle \ ,$$

where $\langle \cdots \rangle$ denotes the subgroup generated by the elements indicated. The class of $a \otimes b$ is denoted $\{a, b\}$, so that $K_2(F)$ is an abelian group (written additively), with generators $\{a, b\}$ for a and b in F^*, and relations

$$\begin{aligned}
&\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\} \\
&\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\} \\
&\{a, 1-a\} = 0 \text{ if } a \text{ is in } F,\ a \neq 0, 1.
\end{aligned}$$

It is a nice exercise to show that these relations imply that $\{a, b\} = -\{b, a\}$ and $\{c, -c\} = 0$ for a, b and c in F^*.

Now consider a (non-singular, projective, geometrically irreducible) curve C defined over $\mathbb{Q}$. Associated to C are K-groups $K_n(C)$ whose definition is a little complicated, but for this paper we need only a certain quotient group[2] $K_2^T(C)$ of $K_2(C)$ ("T" for "tame"), which can be described in a simpler way. Set

$$K_2^T(C) = \ker\left(K_2(F) \xrightarrow{T} \bigoplus_{x \in C(\overline{\mathbb{Q}})} \overline{\mathbb{Q}}^* \right) ,$$

where $F = \mathbb{Q}(C)$ is the function field of C and where the x-component of the map T is the *tame symbol* at x, defined by

$$T_x : \{a, b\} \mapsto (-1)^{\mathrm{ord}_x(a)\mathrm{ord}_x(b)} \frac{a^{\mathrm{ord}_x(b)}}{b^{\mathrm{ord}_x(a)}}(x) \ . \tag{3.1}$$

Note that this definition makes sense since $\dfrac{a^{\mathrm{ord}_x(b)}}{b^{\mathrm{ord}_x(a)}}$ has order zero at x and hence is defined and non-zero at x. It is clear that T_x is a map on $F^* \otimes_{\mathbb{Z}} F^*$, and checking that it is trivial on the symbols $\{a, 1-a\}$ for $a \neq 0, 1$ is a good exercise (which also explains why we want to have the power of -1 in the formula), so T_x defines a map on $K_2(F)$.

We note that if α is an element of $K_2(\mathbb{Q}(C))$, then

$$\prod_{x \in C(\overline{\mathbb{Q}})} T_x(\alpha) = 1 \ , \tag{3.2}$$

a result known as the *product formula* (see [1, Theorem 8.2]).

Beilinson, generalizing work by Bloch [2], defined regulators of the K-groups of C (see [15]). We will describe these in elementary terms for $K_2^T(C)$.

We start with a map from $F^* \times F^*$ to the group of almost everywhere defined 1-forms on the Riemann surface $X = C(\mathbb{C})$ by putting

$$\eta(a, b) = \log|a| \, d \arg b - \log|b| \, d \arg a \ , \tag{3.3}$$

where $\arg a$ is the argument of a. Note that this is well defined (the argument arg is defined up to multiples of 2π, but these map to zero under d) and is a smooth (indeed, real-analytic) 1-form on the complement of the set of zeros and poles of a and b. It is clear that $\eta(a_1 a_2, b) = \eta(a_1, b) + \eta(a_2, b)$ and $\eta(a, b_1 b_2) =$

[1] The actual definition of K_2 (of arbitrary rings) is more complicated (see [13, §5]), and its equivalence for fields with the definition in terms of the "symbols" $\{a, b\}$ is a famous theorem of Matsumoto (see [13, Theorem 11.1]).

[2] usually denoted $H^0(C, \underline{K}_2)$

$\eta(a, b_1) + \eta(a, b_2)$, so that η induces a map (still denoted η) on $F^* \otimes_{\mathbb{Z}} F^*$. Also, $\eta(a, b)$ is closed, since $d\eta(a, b) = \operatorname{Im}(d \log a \wedge d \log b) = 0$.

For any smooth closed 1-form ω defined on the complement of a finite set $S \subset X$, and any smooth oriented loop γ in $X \setminus S$, we have a pairing

$$(\gamma, \omega) = \frac{1}{2\pi} \int_\gamma \omega$$

which depends only on the homology class of γ in $X \setminus S$. As γ moves across a point x in S, the value of (γ, ω) jumps by (C_x, ω), where C_x denotes a small circle around x. A simple calculation shows that $(C_x, \eta(a, b)) = \log |T_x(\{a, b\})|$. It follows that if $\alpha = \sum_i a_i \otimes b_i$ is an element of $F^* \otimes_{\mathbb{Z}} F^*$ such that $T_x(\alpha) = 1$ for all x in X, then $(\,\cdot\,, \eta(\alpha))$ is a well defined map from $H_1(X; \mathbb{Z})$ to $\mathbb{R}$.

Next, one has to check that this map vanishes if $\alpha = a \otimes (1 - a)$ and hence gives us a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : H_1(X; \mathbb{Z}) \times K_2^T(C) \to \mathbb{R}$$

given by $\langle \gamma, \alpha \rangle = (\gamma, \eta(\alpha))$. This follows from the fact that $\eta(a, 1-a) = d\,D(a)$, where $D(z)$ is the Bloch-Wigner dilogarithm function.

Finally, we observe that, if c denotes complex conjugation on X, then $c^*(\eta(\alpha)) = -\eta(\alpha)$ for any α in $K_2^T(C)$, since $c^*(\log |a|) = \log |a|$ and $c^*(d \arg b) = -d \arg b$: a and b are in $\mathbb{Q}(C)^*$, so $(c^* b)(z) = b(\overline{z}) = \overline{b(z)}$. This means that if γ is in $H_1(X; \mathbb{Z})^+$, the c-invariant part of $H_1(X; \mathbb{Z})$, then $\langle \gamma, \alpha \rangle = 0$ for any α in $K_2^T(C)$:

$$\langle \gamma, \alpha \rangle = \frac{1}{2\pi} \int_\gamma \eta(\alpha) = \frac{1}{2\pi} \int_{c \circ \gamma} \eta(\alpha) = \frac{1}{2\pi} \int_\gamma c^*(\eta(\alpha)) = -\frac{1}{2\pi} \int_\gamma \eta(\alpha) = -\langle \gamma, \alpha \rangle\ .$$

We therefore have to compute $\langle \gamma, \alpha \rangle$ only for γ in $H_1(X; \mathbb{Z})/H_1(X; \mathbb{Z})^+$. In practice, we can just as well compute it for all γ in $H_1(X; \mathbb{Z})^-$, the anti-invariants in $H_1(X; \mathbb{Z})$ under the action of c, giving us finally the *regulator pairing*

$$\langle\,\cdot\,,\,\cdot\,\rangle : H_1(X; \mathbb{Z})^- \times K_2^T(C)/\text{torsion} \to \mathbb{R}\ . \tag{3.4}$$

It is easy to see that $H_1(X; \mathbb{Z})^-$ has rank g. Beilinson originally conjectured[3] that the rank of $K_2^T(C)/\text{torsion}$ is also equal to g, that the pairing in (3.4) was non-degenerate, and that there was a relation between $L(C, 2)$ and the absolute value of the determinant of the matrix of this pairing with respect to bases of $H_1(X, \mathbb{Z})^-$ and $K_2^T(C)/\text{torsion}$. Unfortunately, this conjecture was wrong, as $K_2^T(C)/\text{torsion}$ can have rank bigger than g already for $g = 1$, as was discovered by Bloch and Grayson in [3]. They found that one should consider a certain subgroup of $K_2^T(C)/\text{torsion}$ defined by an additional condition, which we now proceed to describe.

The extra condition comes from the fact that one should not consider the curve over $\mathbb{Q}$, but instead a model of it over $\mathbb{Z}$. This is analogous to the situation in Section 1, where one has to consider $K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$ instead of $K_1(k) \cong k^*$ in order to get the correct regulator.

So let $\mathcal{C}$ be a regular proper model of C over $\mathbb{Z}$, i.e., a regular, proper, irreducible two-dimensional scheme over $\mathbb{Z}$ such that the generic fiber $\mathcal{C}_{\mathbb{Q}}$ is isomorphic to C (see, e.g., [12, Chapter 10]). For each prime p, let $\mathcal{C}_p$ be the fiber of $\mathcal{C}$ over $\mathbb{F}_p$. For

[3]Strictly speaking, Beilinson conjectured $K_2^T(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ to have dimension g over $\mathbb{Q}$. However, $K_2^T(C)/K_2(\mathbb{Q})$ is expected to be finitely generated by virtue of a conjecture of Bass, and since $K_2(\mathbb{Q})$ is a torsion group, this would imply the equivalence of Beilinson's formulation and our formulation.

each irreducible component $\mathcal{D}$ of the curve $\mathcal{C}_p$, let $\mathbb{F}_p(\mathcal{D})$ denote its field of rational functions over $\mathbb{F}_p$. Then we define

$$K_2^T(\mathcal{C}) = \ker\left(K_2^T(C) \to \bigoplus_{p,\,\mathcal{D}\subseteq\mathcal{C}_p} \mathbb{F}_p(\mathcal{D})^*\right), \tag{3.5}$$

where the map to $\mathbb{F}_p(\mathcal{D})^*$ is given as follows. The order of vanishing along $\mathcal{D}$ gives rise to a discrete valuation on F, $v_{\mathcal{D}}$. The component of the map in (3.5) corresponding to $\mathcal{D}$ is given by the tame symbol corresponding to $\mathcal{D}$,

$$T_{\mathcal{D}} : \{a, b\} \mapsto (-1)^{v_{\mathcal{D}}(a)v_{\mathcal{D}}(b)} \frac{a^{v_{\mathcal{D}}(b)}}{b^{v_{\mathcal{D}}(a)}}(\mathcal{D}), \tag{3.6}$$

in complete analogy with (3.1). Finally, we set

$$K_2(C;\mathbb{Z}) = K_2^T(\mathcal{C})/\text{torsion}, \tag{3.7}$$

a subgroup of $K_2^T(C)$/torsion. This group, sometimes denoted by $K_2(C)_{\mathbb{Z}}$/torsion (cf. [16, § 3.3]), is independent of the choice of the regular proper model $\mathcal{C}$ of C (see [15, page 13]), justifying the notation.

Remark 3.8. We could have defined $K_2^T(\mathcal{C})$ in a single step as

$$K_2^T(\mathcal{C}) = \ker\left(K_2(F) \to \bigoplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^*\right), \tag{3.9}$$

where $\mathcal{D}$ runs through all irreducible curves on $\mathcal{C}$ and $\mathbb{F}(\mathcal{D})$ stands for the residue field at $\mathcal{D}$. Any such $\mathcal{D}$ is either "vertical", in which case it is a component of some $\mathcal{C}_p$ and $T_{\mathcal{D}}$ is the map in (3.6), or else "horizontal", in which case it corresponds to the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit of some x in $C(\overline{\mathbb{Q}})$ and $T_{\mathcal{D}}$ being trivial is equivalent to T_y being trivial for all y in that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit.

We can now restrict the pairing (3.4) to

$$\langle\,\cdot\,,\,\cdot\,\rangle : H_1(X;\mathbb{Z})^- \times K_2(C;\mathbb{Z}) \to \mathbb{R}, \tag{3.10}$$

and formulate our description of Beilinson's conjecture, as modified in accordance with [3], as follows:

Conjecture 3.11. *Let C be a non-singular, projective, geometrically irreducible curve of genus g defined over $\mathbb{Q}$, and let $X = C(\mathbb{C})$. Then:*

(1) *The group $K_2(C;\mathbb{Z})$ is a free abelian group of rank g and the pairing* (3.10) *is non-degenerate;*
(2) *Let R denote the absolute value of the determinant of this pairing with respect to $\mathbb{Z}$-bases of $H_1(X;\mathbb{Z})^-$ and $K_2(C;\mathbb{Z})$, and let $L^*(C,0)$ be defined as in Section 2. Then $L^*(C,0) = QR$ for some non-zero rational number Q.*

Remark 3.12. The definition of $L^*(C,0)$ requires the analytic continuation of $L(C,s)$, but since the analytic continuation and the expected functional equation of $L(C,s)$ would imply that $L^*(C,0)$ is rationally proportional to $\pi^{-2g}L(C,2)$, Beilinson's conjecture could be formulated without any assumptions about the analytic continuation of $L(C,s)$.

Remark 3.13. In practice, the conjecture is rather intractable, as it seems impossible to compute $K_2(C;\mathbb{Z})$ even after tensoring this group with $\mathbb{Q}$. Indeed, we neither can guarantee finding enough elements to generate $K_2(C;\mathbb{Z})$, nor can we necessarily determine the rank of a subgroup generated by finitely many elements,

as we do not know any practical method for determining if a given combination of elements in $F^* \otimes_{\mathbb{Z}} F^*$ can be written as a sum of Steinberg symbols $a \otimes (1-a)$. But we can try to find g elements in $K_2(C;\mathbb{Z})$ and compute R as in the conjecture using those elements rather than a basis of $K_2(C;\mathbb{Z})$. If R is non-zero numerically we can check the relation with $L^*(C,0)$ as in Conjecture 3.11. Also, if we have more than g elements α_j in $K_2(C;\mathbb{Z})$, the conjecture implies that the maps $\langle\,\cdot\,,\alpha_j\rangle : H_1(X;\mathbb{Z})^- \to \mathbb{R}$ should be linearly dependent over $\mathbb{Z}$, and this too can be checked numerically. Both types of verification will be carried out in Section 10.

Remark 3.14. We have restricted ourselves to the statement of Beilinson's conjecture for a curve over $\mathbb{Q}$, but the conjecture can be formulated equally well for any number field k. Suppose that C/k is a (non-singular, projective, geometrically irreducible) curve of genus g. Let $\mathcal{O}_k$ be the ring of integers of k, and let $\mathcal{C}$ be a model of C/k over $\mathcal{O}_k$. Then one defines $K_2^T(\mathcal{C})$ as in (3.9), with the sum over irreducible curves $\mathcal{D}$ in $\mathcal{C}$. Again, for the "horizontal" curves $\mathcal{D}$, the $T_{\mathcal{D}}$ correspond to the T_x for x in $C(\overline{\mathbb{Q}})$, up to conjugation under $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$. Once again, $K_2^T(\mathcal{C})$/torsion is independent of the choice of the model $\mathcal{C}$, and is denoted by $K_2(C;\mathbb{Z})$. One expects $K_2(C;\mathbb{Z}) \cong \mathbb{Z}^{g\,[k:\mathbb{Q}]}$. The main difference is that the Riemann surface involved will no longer be connected. Instead, let X be the Riemann surface obtained from all points in C over $\mathbb{C}$, using all embeddings of k into $\mathbb{C}$. This is a disjoint union of $[k:\mathbb{Q}]$ connected Riemann surfaces, each of genus g. Complex conjugation acts on this either by swapping conjugate pairs of complex embeddings of k, or by acting on the Riemann surface corresponding to a real embedding of k. Then $H_1(X;\mathbb{Z})^- \cong \mathbb{Z}^{g\,[k:\mathbb{Q}]}$, and there is a pairing $H_1(X;\mathbb{Z})^- \times K_2(C;\mathbb{Z}) \to \mathbb{R}$ given by $\langle\gamma,\alpha\rangle = \frac{1}{2\pi}\int_\gamma \eta(\alpha)$. Here $\eta(\alpha)$ is the 1-form on X which for $\alpha = \{a,b\}$ is given on the connected Riemann surface corresponding to $\sigma : k \to \mathbb{C}$ by $\log|a_\sigma| d\arg b_\sigma - \log|b_\sigma| d\arg a_\sigma$. (The subscripts indicate that we consider the functions on the Riemann surface obtained by applying σ to the coefficients involved in a and b.) One again expects $\langle\,\cdot\,,\,\cdot\,\rangle$ to be non-degenerate, defines R to be the absolute value of the determinant of the matrix of $\langle\,\cdot\,,\,\cdot\,\rangle$ with respect to $\mathbb{Z}$-bases of $H_1(X;\mathbb{Z})^-$ and $K_2(C;\mathbb{Z})$, and conjectures that $L^*(C,0) = QR$ for some non-zero rational number Q.

4. Constructing elements of K_2 from torsion divisors

The first problem in testing Conjecture 3.11 is that it is not at all clear how to construct elements of $K_2^T(C)$ on a given curve C over $\mathbb{Q}$, that is, how to produce rational functions f_i, g_i on C such that $\sum_i \{f_i, g_i\}$ has trivial tame symbol at every point of C.

To understand this condition, consider one symbol $\{f,g\}$ in $K_2(\mathbb{Q}(C))$. If $\mathrm{div}(f)$ and $\mathrm{div}(g)$ have disjoint support, then this symbol lies in $K_2^T(C)$ if and only if $f(P)^{\mathrm{ord}_P(g)} = 1$ for every zero or pole P of g and $g(Q)^{\mathrm{ord}_Q(f)} = 1$ for every zero or pole Q of f. Essentially this says that f equals one (or a root of unity, if $|\mathrm{ord}_P(g)| > 1$) whenever g has a zero or a pole, and similarly with f and g interchanged.

To try to satisfy these conditions, it is natural to look at functions which have only very few zeros and poles. The simplest case is given by functions f and g which have only one (multiple) zero and one (multiple) pole. If one of these points is common for f and g, then it turns out that simply renormalizing the functions is sufficient to satisfy the tame symbol conditions. For this we use the product formula (3.2). All of our examples are then based on the following construction.

Construction 4.1. Let $C/\mathbb{Q}$ be a curve. Assume $P_1, P_2, P_3 \in C(\mathbb{Q})$ are distinct points whose pairwise differences are torsion divisors. Thus there are rational functions f_i with

$$\operatorname{div}(f_i) = m_i(P_{i+1}) - m_i(P_{i-1}), \qquad i \in \mathbb{Z}/3\mathbb{Z}\,,$$

where m_i is the order of $(P_{i+1})-(P_{i-1})$ in the divisor group $\operatorname{Pic}^0(C)$. We then define three elements of $K_2(\mathbb{Q}(C))$ by

$$S_i = \Big\{ \frac{f_{i+1}}{f_{i+1}(P_{i+1})}, \frac{f_{i-1}}{f_{i-1}(P_{i-1})} \Big\}, \qquad i \in \mathbb{Z}/3\mathbb{Z}\,.$$

The functions f_i are unique up to constants, so the symbols S_i are uniquely defined by the points P_i. Moreover, they satisfy the tame symbol condition everywhere:

Lemma 4.2. *The S_i are elements of $K_2^T(C)$.*

Proof. The components of S_i are normalized to make the tame symbol trivial at P_{i-1} and P_{i+1}. By the product formula (3.2) it is also trivial at P_i, this being the only other point in the support of the divisors of f_{i-1} and f_{i+1}. □

Next we show that the three elements S_i generate a subgroup of rank at most 1 of $K_2^T(C)$/torsion.

Proposition 4.3. *We keep the notation of Construction 4.1.*

(1) *There is a unique element $\{P_1, P_2, P_3\}$ of $K_2^T(C)$/*torsion *such that in this group*

$$S_i \;=\; \frac{\operatorname{lcm}(m_1, m_2, m_3)}{m_i}\,\{P_1, P_2, P_3\}, \qquad i = 1, 2, 3\,.$$

(2) *The element $\{P_1, P_2, P_3\}$ is unchanged under even permutations and changes sign under odd permutations of the points.*

Proof. (1) Uniqueness is obvious, so we only need to show existence. Replacing the functions f_i by $f_i/f_i(P_i)$ if necessary, we can assume that $f_i(P_i) = 1$, so that $S_i = \{f_{i+1}, f_i\}$. Now let

$$L = \operatorname{lcm}(m_1, m_2) = \operatorname{lcm}(m_1, m_3) = \operatorname{lcm}(m_2, m_3)\,; \tag{4.4}$$

the three least common multiples are equal since every integer which kills both $(P_i)-(P_{i+1})$ and $(P_{i+1})-(P_{i+2})$ in $\operatorname{Pic}^0(C)$ also kills $(P_i)-(P_{i+2})$, which is their sum. It follows that $L = \operatorname{lcm}(m_1, m_2, m_3)$ and that the quotients

$$r_i = L/m_i, \qquad i = 1, 2, 3$$

are pairwise relatively prime. Then

$$\operatorname{div}(f_i^{r_i}) = L(P_{i+1}) - L(P_{i-1}),$$

so the function $f_1^{r_1} f_2^{r_2} f_3^{r_3}$ has trivial divisor, and is therefore a constant:

$$f_1^{r_1} f_2^{r_2} f_3^{r_3} = c\,. \tag{4.5}$$

Next, we have, in $K_2^T(C)$

$$r_1 S_2 = \{f_3, f_1^{r_1}\}, \qquad r_2 S_1 = \{f_2^{r_2}, f_3\} = -\{f_3, f_2^{r_2}\}\,,$$

so

$$r_1 S_2 - r_2 S_1 = \{f_3, f_1^{r_1} f_2^{r_2}\} = \{f_3, c f_3^{-r_3}\} = \{f_3, (-1)^{r_3} c\}\,,$$

because $\{f_3, -f_3\}$ is trivial. Since the left-hand side is in $K_2^T(C)$, so is $\{f_3, (-1)^{r_3}c\}$. This implies that $c^{2m_3}=1$, so $\{f_3, (-1)^{r_3}c\}$ is torsion. Therefore, modulo torsion, for $i, j = 1, 2, 3$,

$$r_i S_j = r_j S_i \ .$$

Now choose integers $\alpha_1, \alpha_2, \alpha_3$ such that $\sum \alpha_i r_i = 1$ and set

$$T = \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 \ .$$

Then

$$r_1 T = \alpha_1 r_1 S_1 + \alpha_2 r_1 S_2 + \alpha_3 r_1 S_3 = \alpha_1 r_1 S_1 + \alpha_2 r_2 S_1 + \alpha_3 r_3 S_1 = S_1,$$

and, similarly, $r_2 T = S_2$ and $r_3 T = S_3$, all modulo torsion. In particular, the subgroup $\langle S_1, S_2, S_3 \rangle$ of $K_2^T(C)/\text{torsion}$ is generated by the class of T alone.

(2) The first statement follows from the construction in part (1). For the second, let f_i be as above, chosen so that $f_i(P_i)=1$.

Take $\tilde{P}_1 = P_2, \tilde{P}_2 = P_1, \tilde{P}_3 = P_3$ and the functions $\tilde{f}_1 = f_2^{-1}, \tilde{f}_2 = f_1^{-1}, \tilde{f}_3 = f_3^{-1}$, so that $\tilde{\alpha}_1 = \alpha_2$, $\tilde{\alpha}_2 = \alpha_1$ and $\tilde{\alpha}_3 = \alpha_3$. The corresponding symbols are

$$\tilde{S}_1 = \{\tilde{f}_2, \tilde{f}_3\} = \{f_1^{-1}, f_3^{-1}\} = -\{f_1, f_3^{-1}\} = \{f_1, f_3\} = -\{f_3, f_1\} = -S_2,$$

and, similarly, $\tilde{S}_2 = \{f_3^{-1}, f_2^{-1}\} = -S_1$ and $\tilde{S}_3 = \{f_2^{-1}, f_1^{-1}\} = -S_3$. Then modulo torsion,

$$\tilde{T} = \tilde{\alpha}_1 \tilde{S}_1 + \tilde{\alpha}_2 \tilde{S}_2 + \tilde{\alpha}_3 \tilde{S}_3 = -\alpha_2 S_2 - \alpha_1 S_1 - \alpha_3 S_3 = -T \ .$$

□

Proposition 4.6. *Let $C/\mathbb{Q}$ be a curve and P_1, P_2, P_3, P_4 be four distinct points in $C(\mathbb{Q})$ such that all $(P_i) - (P_j)$ are torsion divisors. Then the four elements*

$$\{P_1, \ldots, \widehat{P}_i, \ldots, P_4\} \qquad (1 \le i \le 4)$$

are linearly dependent. More precisely, if m_{ij} is the order of $(P_i) - (P_j)$, then

$$\sum_{i=1}^{4} (-1)^i \, c_i \, \{P_1, \ldots, \widehat{P}_i, \ldots, P_4\} = 0 \tag{4.7}$$

holds in $K_2^T(C)/\text{torsion}$ with

$$c_i = \gcd(m_{ij}, m_{ik}, m_{il}) \qquad (\{i,j,k,l\} = \{1,2,3,4\}) \ . \tag{4.8}$$

Proof. Choose f_{ij} with $\operatorname{div}(f_{ij}) = m_{ij}(P_i) - m_{ij}(P_j)$. The discussion surrounding (4.5) shows that

$$\left(\frac{f_{ij}}{f_{ij}(P_k)}\right)^{\frac{M}{m_{ij}}} \left(\frac{f_{jk}}{f_{jk}(P_i)}\right)^{\frac{M}{m_{jk}}} \left(\frac{f_{ki}}{f_{ki}(P_j)}\right)^{\frac{M}{m_{ki}}} = 1$$

if $\{i, j, k\} \subset \{1, 2, 3, 4\}$ and M is any integer divisible by $2 \operatorname{lcm}(m_{ij}, m_{jk}, m_{ik})^2$. Choose such positive M common for all triples $\{i, j, k\}$ and let $F_{ij} = f_{ij}^{M/m_{ij}}$. Then

$$\operatorname{div}(F_{ij}) = M(P_i) - M(P_j) \qquad (\{i, j\} \subset \{1, 2, 3, 4\})$$

and

$$\frac{F_{ij}}{F_{ij}(P_k)} \frac{F_{jk}}{F_{jk}(P_i)} \frac{F_{ki}}{F_{ki}(P_j)} = 1 \qquad (\{i, j, k\} \subset \{1, 2, 3, 4\}) \ . \tag{4.9}$$

Also define elements of $K_2^T(C)$ by

$$S_{i,j,k} = \Big\{ \frac{F_{ki}}{F_{ki}(P_j)}, \frac{F_{ij}}{F_{ij}(P_k)} \Big\} = \Big\{ \frac{F_{ij}}{F_{ij}(P_k)}, \frac{F_{jk}}{F_{jk}(P_i)} \Big\} = \Big\{ \frac{F_{jk}}{F_{jk}(P_i)}, \frac{F_{ki}}{F_{ki}(P_j)} \Big\} .$$

The asserted relation (4.7), but with c_i replaced by

$$c_i' = \frac{\mathrm{lcm}(m_{lj}, m_{lk}, m_{jk})}{m_{lj} m_{lk} m_{jk}} \qquad (\{i,j,k,l\} = \{1,2,3,4\}), \tag{4.10}$$

then follows from the statement that, in $K_2^T(C)$/torsion,

$$S_{1,2,3} - S_{1,2,4} + S_{1,3,4} - S_{2,3,4} = 0 . \tag{4.11}$$

To prove (4.11), let $f = F_{14}/F_{14}(P_2)$, $g = F_{24}/F_{24}(P_3)$ and $h = F_{34}/F_{34}(P_1)$. Rescaling if necessary, we may also assume that $F_{12} = fg^{-1}$, $F_{23} = gh^{-1}$ and $F_{31} = hf^{-1}$. Finally, let

$$\alpha = f(P_3), \;\; \beta = g(P_1), \;\; \gamma = h(P_2) .$$

Now we apply (4.9) for $\{i,j,k\} = \{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3\}$. This gives, respectively, that

$$(fg^{-1})(P_4) = \beta^{-1}, \quad (gh^{-1})(P_4) = \gamma^{-1}, \quad (hf^{-1})(P_4) = \alpha^{-1} \quad \text{and} \quad \alpha\beta\gamma = 1 .$$

Finally, using all of these we expand the symbols $S_{i,j,k}$ in terms of the 8 generators,

$$v = (\{f,g\}, \{f,h\}, \{g,h\}, \{f,\beta\}, \{g,\alpha\}, \{g,\beta\}, \{h,\alpha\}, \{\alpha,\beta\}) .$$

We find

$$\begin{aligned} S_{1,2,3} &= \{fg^{-1}/\alpha, gh^{-1}/\beta\} &&= (1,-1,1,-1,1,1,-1,1)\cdot v , \\ -S_{1,2,4} &= \{f, g/\beta\} &&= (-1,0,0,1,0,0,0,0)\cdot v , \\ S_{1,3,4} &= \{f/\alpha, h\} &&= (0,1,0,0,0,0,1,0)\cdot v , \\ -S_{2,3,4} &= \{g, h/\gamma\} &&= (0,0,-1,0,-1,-1,0,0)\cdot v . \end{aligned}$$

Therefore the left-hand side of (4.11) reduces to one symbol $\{\alpha,\beta\}$. This symbol can be also rewritten in terms of the original functions. One shows that

$$\{\alpha,\beta\} = \pm \Big\{ \Big(\frac{f_{kl}(P_i)}{f_{kl}(P_j)}\Big)^{\frac{M}{m_{kl}}}, \Big(\frac{f_{il}(P_j)}{f_{il}(P_k)}\Big)^{\frac{M}{m_{il}}} \Big\} .$$

for all combinations of four distinct indices i, j, k and l with $|i-k| = |j-l| = 2$. In any case this element comes from K_2 of a number field, which is torsion. Hence the asserted relation holds.

It remains to show that the numbers c_i and c_i' defined by (4.8) and (4.10) are proportional. Equivalently, if $\gamma_i = v_p(c_i)$ and $\gamma_i' = v_p(c_i')$ denote the valuations of c_i and d_i' at some prime p, then we must show that $\gamma_i - \gamma_i'$ is independent of i. The numbers γ_i and γ_i' are given by

$$\gamma_i = \min_{j\neq i}\{\nu_{ij}\}, \qquad \gamma_i' = \max_{j,k\neq i}\{\nu_{jk}\} - \sum_{j,k\neq i,\; j<k} \nu_{jk} \qquad (i = 1,2,3,4), \tag{4.12}$$

where ν_{ij} denotes $v_p(m_{ij})$. If we use the six numbers ν_{ij} to label the edges of a tetrahedron T in the obvious way, then the relation (4.4) says that the two largest labels of the sides of any face of T are equal. From this we find easily that there are two possibilities: if three incident edges of T have the same label, then, possibly after renumbering, we have $\nu_{12} = \nu_{13} = \nu_{14} = a$, $\nu_{23} = \nu_{24} = b$, $\nu_{34} = c$ for some integers $a \ge b \ge c$, while if this does not happen then, again up to renumbering, we have $\nu_{13} = \nu_{14} = \nu_{23} = \nu_{24} = a$, $\nu_{12} = b$, $\nu_{34} = c$ with $a > b$, c. From (4.12) we then find that $(\gamma_1, \ldots, \gamma_4)$ and $(\gamma_1', \ldots, \gamma_4')$ are given by

(a, b, c, c) and $(-b-c, -a-c, -a-b, -a-b)$ in the first case and by (b, b, c, c) and $(-a-c, -a-c, -a-b, -a-b)$ in the second case, so that in both cases $\gamma_i - \gamma_i' = a+b+c$ for all i. This completes the proof of Proposition 4.6. □

Corollary 4.13. *Let C be a curve defined over $\mathbb{Q}$ and $P_1, \dots, P_n \in C(\mathbb{Q})$ points such that all $(P_i) - (P_j)$ are torsion. Then the subspace of $K_2^T(C) \otimes \mathbb{Q}$ generated by all elements $\{P_i, P_j, P_k\}$ is already generated by those of the form $\{P_1, P_i, P_j\}$.*

The corollary implies that the space spanned by the $n(n-1)(n-2)$ symbols $\{P_i, P_j, P_k\}$, which already by part (2) of Proposition 4.3 had dimension at most $\binom{n}{3}$, in fact has dimension at most $\binom{n-1}{2}$.

Remark 4.14. If $C/\mathbb{Q}$ is a curve, it is sometimes convenient to consider points P_i in $C(\overline{\mathbb{Q}})$. Then we have to work in $K_2^T(C_{\overline{\mathbb{Q}}})$, which is defined as the kernel of the tame symbol (given by (3.1)),

$$K_2^T(C_{\overline{\mathbb{Q}}}) = \ker\left(K_2(\overline{\mathbb{Q}}(C)) \to \bigoplus_{x \in C(\overline{\mathbb{Q}})} \overline{\mathbb{Q}}^* \right).$$

The product formula (3.2) still holds for elements in $K_2(\overline{\mathbb{Q}}(C))$. All the results in this section remain true in this context. Moreover, since the inclusion of $\mathbb{Q}(C)$ into $\overline{\mathbb{Q}}(C)$ induces a map from $K_2(\mathbb{Q}(C))$ to $K_2(\overline{\mathbb{Q}}(C))$ and the tame symbol on both groups is given by the same formula, we can check if an element in $K_2(\mathbb{Q}(C))$ lies in $K_2^T(C)$ using the tame symbol on its image in $K_2(\overline{\mathbb{Q}}(C))$.

In fact, the map $K_2(\mathbb{Q}(C)) \to K_2(\overline{\mathbb{Q}}(C))$ has torsion kernel and we may identify $K_2(\mathbb{Q}(C)) \otimes \mathbb{Q}$ with the subspace of $K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially. The same statements hold when we replace $K_2(\mathbb{Q}(C))$ by $K_2^T(C)$ and $K_2(\overline{\mathbb{Q}}(C))$ by $K_2^T(C_{\overline{\mathbb{Q}}})$

The final result of this section, a strengthening of Corollary 4.13, says that we cannot construct any more elements of $K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ using only functions whose zeros and poles differ by torsion divisors than those already given by Construction 4.1.

Proposition 4.15. *Let $S \subseteq C(\overline{\mathbb{Q}})$ and $P_0 \in S$ be such that $(P) - (P_0)$ is a torsion divisor for all P in S. Let V be the subspace of $K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ generated by all elements $\{P, Q, P_0\}$ with $P, Q \in S$ and W the subspace of $K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ generated by all symbols $\{f, g\}$ with $\operatorname{div}(f)$ and $\operatorname{div}(g)$ supported in S. Then*

$$W \cap K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = V.$$

Proof. Let $\xi = \sum_i \{f_i, g_i\}$ be an element of $W \cap K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$. Replacing S with the finite set of all zeros and poles of the f_i and g_i, by assumption there exist an integer $N > 0$ and, for each $P \in S \setminus \{P_0\}$, a function $f_P \in \overline{\mathbb{Q}}(C)^*$ with $\operatorname{div}(f_P) = N(P) - N(P_0)$. Then each f_i^N or g_i^N is, up to a scalar factor, a multiplicative combination of the functions f_P, so $N^2\xi$ is a linear combination of symbols of the form $\{f_P, f_Q\}$, $\{f_P, c\}$ and $\{c, c'\}$ with $c,\, c' \in \overline{\mathbb{Q}}^*$. But $\{c, c'\} = 0$ in $K_2^T(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ and $\{f_P, f_Q\}$ is the sum of $\{f_P, f_Q(P)\} + \{f_P(Q), f_Q\}$ and a multiple of $\{P, Q, P_0\} \in V$, so ξ can be written as $\sum_P \{f_P, c_P\} + \xi'$ with $\xi' \in V$ for some constants $c_P \in \overline{\mathbb{Q}}^*$. Since the tame symbol of $\{f_P, c_P\}$ at x is c_P^{-N} for $x = P$ and trivial for all other $x \in C(\overline{\mathbb{Q}}) \setminus \{P_0\}$, the fact that some multiple of ξ (and hence also of $\xi - \xi'$) has trivial tame symbol everywhere implies that each c_P is a root of unity. Hence $\xi \in V$. □

5. Torsion divisors on hyperelliptic curves

Now suppose that C is a hyperelliptic curve of genus $g \geq 1$ defined over $\mathbb{Q}$. The hyperelliptic involution on C determines a double cover $\phi : C \to \mathbb{P}^1$ ramified at $2g+2$ points, the fixed points of the involution. Assume that one of these points ∞_C is defined over $\mathbb{Q}$. After a change of coordinates on $\mathbb{P}^1$ we can assume that $\phi(\infty_C) = \infty_{\mathbb{P}^1}$. Then C gets a model

$$y^2 = c_{2g+1}x^{2g+1} + c_{2g}x^{2g} + \ldots + c_1 x + c_0, \qquad c_{2g+1} \neq 0, \tag{5.1}$$

where the polynomial on the right has coefficients in $\mathbb{Q}$ and has no multiple roots. This equation can be seen either as defining a double cover of $\mathbb{P}^1$ or as a curve (singular for $g > 1$) in $\mathbb{P}^2$ whose normalization is C. The point $\infty = \infty_C$ is then the unique point at infinity of this normalization. The cover ϕ is given by the function x on C and its ramification points are ∞ and the $T_\alpha = (\alpha, 0)$, where α runs through the roots of the right-hand side of (5.1) in $\overline{\mathbb{Q}}$.

We will look for points P such that the divisor $(P) - (\infty)$ is m-torsion for some m. Such points will be called $(m\text{-})$*torsion points*. Of course, if P, Q are torsion points, then $(P) - (Q)$ is also a torsion divisor. If we succeed in constructing many torsion points P_i, then we get many elements $\{\infty, P_i, P_j\} \in K_2^T(C)/\text{torsion}$ by using Proposition 4.3. Here are some examples of curves with torsion points:

Example 5.2: 2-torsion. With notation as before, any non-trivial difference $(P) - (Q)$ for P and Q among ∞ and the T_α is 2-torsion:

$$2(T_\alpha) - 2(T_\beta) = \operatorname{div}\Big(\frac{x-\alpha}{x-\beta}\Big), \qquad 2(T_\alpha) - 2(\infty) = \operatorname{div}(x-\alpha)\,.$$

So the T_α are 2-torsion points, though not necessarily defined over $\mathbb{Q}$.

Conversely, suppose given an arbitrary curve C over $\mathbb{Q}$ and two distinct points $P, Q \in C(\mathbb{Q})$ such that the divisor $(P) - (Q)$ is 2-torsion. Say $\operatorname{div}(\phi) = 2(P) - 2(Q)$ with ϕ defined over $\mathbb{Q}$. Then $\phi : C \to \mathbb{P}^1$ is a double cover defined over $\mathbb{Q}$. It follows that C is hyperelliptic and admits a model (5.1) with $Q = \infty$, P among the T_α.

Unfortunately, we cannot use 2-torsion points alone to obtain interesting elements in $K_2^T(C_{\overline{\mathbb{Q}}})/\text{torsion}$ by the construction of Section 4: the calculation

$$\Big\{\frac{x-\beta}{\alpha-\beta}, \frac{\beta-\alpha}{x-\alpha}\Big\} = \Big\{\frac{x-\alpha}{\beta-\alpha}, \frac{x-\beta}{\alpha-\beta}\Big\} = \Big\{\frac{x-\alpha}{\beta-\alpha}, 1 - \frac{x-\alpha}{\beta-\alpha}\Big\} = 0$$

shows that $\{\infty, T_\alpha, T_\beta\}$ is trivial for any α and β, and Proposition 4.6 (or rather, its analogue over $\overline{\mathbb{Q}}$) then shows that the elements $\{T_\alpha, T_\beta, T_\gamma\}$ also vanish. The 2-torsion points can nevertheless be used, but only in combination with the torsion points of other orders which we describe next.

Example 5.3: (2g+1)-torsion. Assume that the hyperelliptic equation (5.1) has the special form

$$y^2 = c\,x^{2g+1} + (b_g x^g + \ldots + b_1 x + b_0)^2 \tag{5.4}$$

for some $c \neq 0$. We can scale x and y to make $c = -1$ if desired, which allows a uniform treatment together with Example 5.6 below. The substitution $y \mapsto y + \sum b_i x^i$ transforms this equation to

$$y^2 + 2(b_g x^g + \ldots + b_1 x + b_0)y = cx^{2g+1}\,. \tag{5.5}$$

The point $O = (0, 0)$ (corresponding to $(0, b_0)$ on the curve (5.4)) lies on this curve and the function y has divisor $(2g+1)(O)-(2g+1)(\infty)$, so O is a $(2g+1)$-torsion point.

In fact, any curve C as in (5.1) with a non-trivial rational $(2g+1)$-torsion point O is isomorphic to a curve of the form (5.4) and hence (5.5). Indeed, given such a curve C, consider the local system

$$L = H^0(C, \mathcal{O}_C((2g+1)\infty)).$$

By Riemann-Roch, the dimension of this as $\mathbb{Q}$-vector space is $1-g+2g+1 = g+2$. It is easy to see that $\{1, x, \ldots, x^g, y\}$ is a basis. So for a point $O = (x_O, y_O) \in C(\mathbb{Q})$ the divisor $(2g+1)(O) - (2g+1)(\infty)$ is principal if and only if there is an element of L,

$$h(x, y) = y - \tilde{b}_g x^g - \ldots - \tilde{b}_1 x - \tilde{b}_0 ,$$

which vanishes to order exactly $2g+1$ at O. (Note that necessarily it cannot be expressed in terms of x alone because the order of such a function is at most $2g$.) Then the curve defined by $h(x, y) = 0$ can have only the point (x_O, y_O) in common with C, and after substituting $y = \tilde{b}_g x^g + \ldots + \tilde{b}_1 x + \tilde{b}_0$ in (5.1), we see that the equation of the curve is of the form

$$y^2 = c(x - x_O)^{2g+1} + (\tilde{b}_g x^g + \ldots + \tilde{b}_1 x + \tilde{b}_0)^2 .$$

After a translation we can assume $x_O = 0$ and the equation of the curve becomes of the form (5.4) with $O = (0, b_0)$.

Example 5.6: (2g+2)-torsion. Similar to the previous example, a curve

$$y^2 = -c^2 x^{2g+2} + (cx^{g+1} + b_g x^g + \ldots + b_1 x + b_0)^2 ,$$

with $b_g, c \neq 0$ is of the form (5.1) and it has a model

$$y^2 + 2(cx^{g+1} + b_g x^g + \ldots + b_1 x + b_0)y = -c^2 x^{2g+2} .$$

In this model $\operatorname{div}(y) = (2g+2)(O)-(2g+2)(\infty)$, so the curve has a $(2g+2)$-torsion point $O = (0, 0)$. Scaling y we can assume that $c = 1$, hence $-c^2 = -1$, allowing a uniform treatment together with Example 5.3.

As in Example 5.3, one can use the linear system

$$L = H^0(C, \mathcal{O}_C((2g+2)\infty))$$

to show that any curve (5.1) with a rational $(2g+2)$-torsion point has a model of this form.

6. Elements of K_2 for hyperelliptic curves

In this section we study elements of $K_2^T(C)$ constructed from torsion points on a non-singular hyperelliptic curve $C/\mathbb{Q}$ of genus g. We show that using only 2-torsion points is not sufficient to construct non-torsion elements. So we will use curves which also have rational torsion points of order $2g+1$ or $2g+2$.

As discussed in Examples 5.3 and 5.6, such a curve $C/\mathbb{Q}$ can always be given by an equation of the form

$$y^2 + f(x)\, y + x^d = 0 \tag{6.1}$$

in one of the following two cases:

$$\text{(6.2)} \quad \begin{array}{lll} \text{(a)} & d = 2g+1, & f(x) = b_g x^g + \ldots + b_1 x + b_0 ; \\ \text{(b)} & d = 2g+2, & f(x) = 2x^{g+1} + b_g x^g + \ldots + b_1 x + b_0 \text{ and } b_g \neq 0 . \end{array}$$

This curve is isomorphic to $y^2 = t(x)$, where $t(x)$ is the 2-torsion polynomial,

$$t(x) = -x^d + f(x)^2/4\,.$$

Since we want C to be non-singular, we assume that $t(x)$ has no multiple roots, and, in particular, that $b_0 \neq 0$.

On the curve (6.1) we have a rational point $O = (0, 0)$ and the divisor $d(O)$–$d(\infty)$ is principal, namely equal to $\operatorname{div}(y)$. There is also a "reflected" d-torsion point

$$O' = (0, -f(0)),$$

the image of O under the hyperelliptic involution $(x, y) \mapsto (x, -y - f(x))$.

Moreover, every rational root α of $t(x)$ gives a rational 2-torsion point $T_\alpha = (\alpha, -f(\alpha)/2)$. All the pairwise differences of the points O, O', ∞ and the T_α are torsion divisors. So the results of Section 4 apply and we get elements of $K_2^T(C)/\text{torsion}$, such as

$$\{\infty, O, T_\alpha\}, \quad \{\infty, O, O'\}, \quad \{T_\alpha, T_\beta, T_\gamma\}, \quad \dots \quad .$$

If we replace $K_2^T(C)$ with $K_2^T(C_{\overline{\mathbb{Q}}})$ then, by Remark 4.14, we can also use points T_α for any root $\alpha \in \overline{\mathbb{Q}}$ of the 2-torsion polynomial $t(x)$, not necessarily rational. The following proposition gives relations in $K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ among these elements. We mention that the two relations involving $\{\infty, O, O'\}$ were first observed in computer calculations at the regulator level.

Proposition 6.3. *Assume $g \geq 2$, and let V be the $\mathbb{Q}$-subspace of $K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ generated by the elements of the form $\{P_i, P_j, P_k\}$ where P_i, P_j, P_k run through the points ∞, O, O' and the T_α for $\alpha \in \overline{\mathbb{Q}}$ a root of $t(x)$. Then V is already generated by the elements of the form $\{\infty, O, T_\alpha\}$. More precisely, if we write $d = 2g + \varepsilon$ with $\varepsilon = 1$ or 2, then we have the relations*

$$\{\infty, T_\alpha, T_\beta\} = 0\,,$$
$$\{\infty, O, T_\alpha\} + \{\infty, O', T_\alpha\} = 0$$

and

$$\varepsilon\{\infty, O, O'\} = \sum_\alpha \{\infty, O, T_\alpha\}\,,$$

together with the symmetry properties of Proposition 4.3 and the tetrahedron relation of Proposition 4.6.

Furthermore, if $f(0)^2 = 1$, then

$$\{-f(0)y, -x\}$$

is in $K_2^T(C)$, and its class $\mathbb{M}$ in $K_2^T(C)/\text{torsion}$ satisfies

$$d\,\mathbb{M} = \varepsilon\{\infty, O, O'\}\,.$$

Proof. The proof of the first relation was already given in Example 5.2. The other three relations involve $\{\infty, O, T_\alpha\}$, $\{\infty, O', T_\alpha\}$ and $\{\infty, O, O'\}$. To make these elements explicit from the definitions in Construction 4.1 and Proposition 4.3 we need to determine the order d' of $(O) - (O') = 2(O) - 2(\infty)$ and the order d''_α of $(O) - (T_\alpha) = (T_\alpha) - (O')$ in $\operatorname{Pic}^0(C)$.

In fact, we have

$$d' = d/\varepsilon \qquad \text{and} \qquad d''_\alpha = 2d/\varepsilon\,.$$

Namely, if $d = 2g+1$ then $d' = d$ and if $d = 2g+2$ then $d' = d/2$. And if $d = 2g+1$, then $(O)-(T_\alpha) = [(O)-(\infty)] - [(T_\alpha)-(\infty)]$ must have order $d''_\alpha = 2d$, but if $d = 2g+2$ the order can be d or $d/2$. However, it can be $d/2 = g+1$ only if $(g+1)[(O)-(\infty)] = (T_\alpha)-(\infty) = (\infty)-(T_\alpha)$, so that $(g+1)(O)+(T_\alpha)-(g+2)(\infty)$ is the divisor of some non-zero h in $H^0(C, \mathcal{O}_C((g+2)\infty))$. Since $g \geq 2$, $g+2 \leq 2g$, so h lies in $H^0(C, \mathcal{O}_C(2g\infty))$, which has basis $\{1, x, \ldots, x^g\}$ (cf. Example 5.3). But that implies that $\mathrm{div}(h)$ is invariant under the hyperelliptic involution whereas $(g+1)(O)+(T_\alpha)-(g+2)(\infty)$ is not because $O \neq O'$. So $d''_\alpha = d/2$ is impossible.

Using those values of d' and d''_α in the definitions we now see that

$$\begin{aligned}
\{\infty, O, T_\alpha\} &= \left\{ \frac{y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha} \right\}, \\
\{\infty, O', T_\alpha\} &= \left\{ \frac{-f(x)-y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha} \right\}
\end{aligned} \tag{6.4}$$

and

$$\varepsilon\{\infty, O, O'\} = \left\{ \frac{y}{-f(0)}, \frac{y+f(x)}{f(0)} \right\},$$

where we write elements in $K_2^T(C_{\overline{\mathbb{Q}}})$ for their classes in $K_2^T(C_{\overline{\mathbb{Q}}})$/torsion.

The second relation in the proposition now follows from

$$\begin{aligned}
\left\{\frac{y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha}\right\} + \left\{\frac{-f(x)-y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha}\right\} &= \left\{\frac{y(f(x)+y)}{-f(\alpha)^2/4}, 1-\frac{x}{\alpha}\right\} \\
&= \left\{\frac{-x^d}{-\alpha^d}, 1-\frac{x}{\alpha}\right\} = d\left\{\frac{x}{\alpha}, 1-\frac{x}{\alpha}\right\} = 0\,.
\end{aligned}$$

In order to prove the third relation, we notice that

$$\begin{aligned}
2\left\{ \frac{y}{-f(\alpha)/2}\,, \frac{x-\alpha}{-\alpha} \right\} &= \left\{ \frac{y^2}{f(\alpha)^2/4}\,, 1-\frac{x}{\alpha} \right\} = \left\{ \frac{-y^2}{-\alpha^d}\,, 1-\frac{x}{\alpha} \right\} \\
&= \left\{ \frac{y^2}{\alpha^d}\,, 1-\frac{x}{\alpha} \right\} - d\left\{ \frac{x}{\alpha}\,, 1-\frac{x}{\alpha} \right\} = \left\{ \frac{y^2}{x^d}\,, 1-\frac{x}{\alpha} \right\},
\end{aligned} \tag{6.5}$$

so that taking the sum over all roots $\alpha \in \overline{\mathbb{Q}}$ of $t(x)$ gives

$$\begin{aligned}
\sum_{t(\alpha)=0} \left\{ \frac{y^2}{x^d}, 1-\frac{x}{\alpha} \right\} &= \left\{ \frac{y^2}{x^d}, \prod_{t(\alpha)=0} \left(1-\frac{x}{\alpha}\right) \right\} \\
&= \left\{ \frac{y^2}{x^d}, \frac{t(x)}{t(0)} \right\} = \left\{ \frac{y^2}{x^d}, \frac{(y+f(x)/2)^2}{f(0)^2/4} \right\},
\end{aligned} \tag{6.6}$$

where we used that $t(x) = (y+f(x)/2)^2$. Also, already in $K_2^T(C)$,

$$\begin{aligned}
2\left\{ \frac{y}{-f(0)}, \frac{y+f(x)}{f(0)} \right\} &= 2\left\{ \frac{y}{-f(0)}, \frac{y\,(y+f(x))}{f(0)^2} \right\} \\
&= \left\{ \frac{y^2}{f(0)^2}, \frac{-x^d}{f(0)^2} \right\} = \left\{ \frac{y^2}{x^d}, \frac{-x^d}{f(0)^2} \right\},
\end{aligned} \tag{6.7}$$

so that $2\varepsilon\{\infty, O, O'\} - 2\sum_\alpha \{\infty, O, T_\alpha\}$ is equal to the class in $K_2^T(C_{\overline{\mathbb{Q}}})$/torsion of

$$\left\{ \frac{y^2}{x^d}, \frac{-x^d}{f(0)^2} \right\} - \left\{ \frac{y^2}{x^d}, \frac{(y+f(x)/2)^2}{f(0)^2/4} \right\} = \left\{ \frac{y^2}{x^d}, \frac{-x^d}{(2y+f(x))^2} \right\}. \tag{6.8}$$

That this last element is zero in $K_2^T(C_{\overline{\mathbb{Q}}})$ can be seen by applying the identity

$$\Big\{ \frac{a}{1-a}, \frac{a(a-1)}{(1-2a)^2} \Big\} = \Big\{ \Big(\frac{1-a}{a}\Big)^2, 1-\Big(\frac{1-a}{a}\Big)^2 \Big\} - \Big\{ \frac{a}{1-a}, -\frac{a}{1-a} \Big\} \\ - 2\Big\{ a, -a \Big\} + 2\Big\{ 1-a, a \Big\} = 0\,,$$

valid in K_2 of any field if a, $1-a$ and $1-2a$ are non-zero, to $a = -y/f(x)$, so that $1-a = -x^d/f(x)y$ and $1-2a = (2y+f(x))/f(x)$. This comples the proof of the third relation.

Finally, if $f(0)^2 = 1$, then all the tame symbols T_P of $\{-f(0)y, -x\}$ are trivial for P different from $O = (0,0)$, $O' = (0, -f(0))$ and ∞, and for those three points we find that

$$\begin{aligned} T_{(0,0)}(\{-f(0)y,-x\}) &= (-1)^d \frac{-f(0)y}{(-x)^d}\Big|_{(0,0)} = 1\,, \\ T_{(0,-f(0))}(\{-f(0)y,-x\}) &= (-1)^0 \frac{-f(0)y}{(-x)^0}\Big|_{(0,-f(0))} = f(0)^2 = 1\,, \\ T_\infty(\{-f(0)y,-x\}) &= 1\,. \end{aligned}$$

The first equation uses that $y(y+f(x)) = -x^d$, so that the function $-y/x^d$ equals $1/(y+f(x))$ and therefore assumes the value $1/f(0)$ at $(0,0)$. The third equation follows from the first two and the product formula. Then (6.7) shows that

$$2d\{-f(0)y, -x\} = \{y^2, -x^d\} = 2\Big\{ \frac{y}{-f(0)}, \frac{y+f(x)}{f(0)} \Big\}$$

in $K_2^T(C)$, so that $2d\mathbb{M} = 2\varepsilon\{\infty, O, O'\}$ in $K_2^T(C)$/torsion. This finishes the proof. $\square$

Remark 6.9. When $g = 1$, the statements of Proposition 6.3 still hold, but when $d = 2g+2 = 4$, the identity $\varepsilon\{\infty, O, O'\} = \sum_\alpha \{\infty, O, T_\alpha\}$ is replaced by

$$2\{\infty, O, O'\} = \{\infty, O, T_\alpha\} + \{\infty, O, T_{\alpha'}\} + 2\{\infty, O, T_\beta\}\,,$$

where T_α, $T_{\alpha'}$ and T_β are the points of order 2, and $2(O) - 2(\infty) = (T_\beta) - (\infty)$ in $\mathrm{Pic}^0(C_{\overline{\mathbb{Q}}})$. The proof is the same, using that $2\{\infty, O, O'\} = \{\frac{y}{-f(0)}, \frac{y+f(x)}{f(0)}\}$, $\{\infty, O, T_\gamma\} = \{\frac{y}{-f(\gamma)/2}, \frac{x-\gamma}{-\gamma}\}$ for $\gamma = \alpha$ or α' but $2\{\infty, O, T_\beta\} = \{\frac{y}{-f(\beta)/2}, \frac{x-\beta}{-\beta}\}$, and similarly for O' instead of O in the last two identities. (Here again we write elements in $K_2^T(C_{\overline{\mathbb{Q}}})$ for their classes in $K_2^T(C_{\overline{\mathbb{Q}}})$/torsion.) In (6.8) one takes $4\{\infty, O, O'\} - 2\{\infty, O, T_\alpha\} - 2\{\infty, O, T_{\alpha'}\} - 4\{\infty, O, T_\beta\}$ as the left-hand side, whereas the right-hand side vanishes as before.

Now we return from $K_2^T(C_{\overline{\mathbb{Q}}})$/torsion to $K_2^T(C)$/torsion, the group that we are interested in. Here we have elements $\{\infty, O, T_\alpha\}$, where α is a rational root of $t(x)$ but, in fact, we have more. As made explicit in (6.4), $\{\infty, O, T_\alpha\}$ is the class of an element of $K_2^T(C)$, namely $\{\frac{y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha}\}$. (If $g = 1$ and $d = 4$ this statement has to be slightly modified according to Remark 6.9.) If $m(x)$ is a rational factor of the 2-torsion polynomial $t(x)$, then the element $\sum_{m(\alpha)=0}\{\frac{y}{-f(\alpha)/2}, \frac{x-\alpha}{-\alpha}\}$ (where the sum is taken over the roots of $m(x)$ in $\overline{\mathbb{Q}}$) in $K_2^T(C_{\overline{\mathbb{Q}}})$ can be shown to come from $K_2^T(C)$. But if we multiply it by 2 and use (6.5), then, by a calculation similar

to (6.6), we get explicitly that

$$2 \sum_{m(\alpha)=0} \left\{ \frac{y}{-f(\alpha)/2}, 1 - \frac{x}{\alpha} \right\} = \left\{ \frac{y^2}{x^d}, \frac{m(x)}{m(0)} \right\} . \tag{6.10}$$

This computation shows that we get an element of $K_2^T(C)$ for every rational irreducible factor $m(x)$ of the 2-torsion polynomial $t(x)$, not just for the linear ones. Thus we get k explicit elements of $K_2^T(C)/\text{torsion}$, where k is the number of irreducible rational factors of the 2-torsion polynomial of the curve in (6.1). This gives a map from $\mathbb{Z}^k$ to $K_2^T(C)/\text{torsion}$. In summary, we have:

Construction 6.11. Let $C/\mathbb{Q}$ be given by (6.1), (6.2). Let $m_1, \ldots, m_k$ be the irreducible factors in $\mathbb{Q}[x]$ (up to multiplication by $\mathbb{Q}^*$) of the 2-torsion polynomial $t(x) = -x^d + f(x)^2/4$. To each of them we associate an element of $K_2^T(C)/\text{torsion}$,

$$M_j = \text{the class of } \left\{ \frac{y^2}{x^d}, \frac{m_j(x)}{m_j(0)} \right\}, \qquad 1 \le j \le k \, . \tag{6.12}$$

Extending by linearity gives a map

$$\begin{array}{ccl} \mathbb{Z}^k & \longrightarrow & K_2^T(C)/\text{torsion} \\ (n_1, \ldots, n_k) & \longmapsto & \displaystyle\sum_{j=1}^k n_j M_j = \text{the class of } \left\{ \frac{y^2}{x^d}, \prod_{j=1}^k \frac{m_j(x)^{n_j}}{m_j(0)^{n_j}} \right\} . \end{array} \tag{6.13}$$

In two special cases, the statement of Construction 6.11 can be refined a little. One case is when $f(0)^2 = 1$, when we can use the class $\mathbb{M}$ of Proposition 6.3 to replace the $\mathbb{Z}^k$ in (6.13) by a larger lattice of the same rank. The other situation occurs only if $d = 2g + 2$, when it turns out that there is always a relation among the M_j (also originally discovered during calculations of the regulators) and hence the rank of the image is at most $k - 1$.

Proposition 6.14. *Let all notation be as in Construction 6.11.*

(1) *If $f(0) = \pm 1$, then the class $\mathbb{M}$ in $K_2^T(C)/\text{torsion}$ of $\{-f(0)y, -x\}$ satisfies*

$$2d\mathbb{M} = \sum_{j=1}^k M_j \, .$$

(2) *If $d = 2g + 2$, so that $4t(x) = (f(x) - 2x^{g+1})(f(x) + 2x^{g+1})$, and*

$$f(x) - 2x^{g+1} = m_1 \ldots m_l \, , \qquad f(x) + 2x^{g+1} = m_{l+1} \ldots m_k$$

with m_j irreducible, then the corresponding classes M_j satisfy

$$M_1 + \ldots + M_l = M_{l+1} + \ldots + M_k \, .$$

Proof. Relation (1) follows from the two identities involving $\varepsilon\{\infty, O, O'\}$ in Proposition 6.3, (6.4) and (6.10) (or the corresponding statements in Remark 6.9 if $g = 1$ and $d = 4$).

In order to prove (2), let $H = -y/x^{g+1}$, so that $H + 1/H = f(x)/x^{g+1}$. Then the relation follows immediately from the fact that, in $K_2^T(C)$,

$$\begin{aligned} \left\{ \frac{y^2}{x^d}, \frac{f(x) - 2x^{g+1}}{f(x) + 2x^{g+1}} \right\} &= \left\{ H^2, \frac{H - 2 + 1/H}{H + 2 + 1/H} \right\} = 2 \left\{ H, \frac{(1-H)^2}{(1+H)^2} \right\} \\ &= 4\{H, 1 - H\} - 4\{-H, 1 - (-H)\} = 0 \, . \end{aligned}$$

□

Remark 6.15. If we identify $K_2(\mathbb{Q}(C)) \otimes \mathbb{Q}$ with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariants of $K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ as in Remark 4.14, and let $W \subseteq K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ be as in Proposition 4.15 (with $S = \{\infty, O, O', T_{\alpha_1}, \dots, T_{\alpha_{2g+1}}\}$, where $\alpha_1, \dots, \alpha_{2g+1}$ are the roots of $t(x)$ in $\overline{\mathbb{Q}}$, and ∞ playing the role of P_0), then we have that

$$W \cap K_2^T(C) \otimes \mathbb{Q} = \langle M_1, \dots, M_k \rangle_{\mathbb{Q}} ,$$

where the M_j are as in Construction 6.11, and $\langle \cdots \rangle_{\mathbb{Q}}$ indicates the $\mathbb{Q}$-vector space they generate. Therefore the M_j's give us everything in $K_2^T(C) \otimes \mathbb{Q}$ that we can get from combinations of symbols $\{f, g\}$ where f and g are in $\overline{\mathbb{Q}}(C)^*$ and $\mathrm{div}(f)$ and $\mathrm{div}(g)$ are supported in $\{\infty, O, O', T_{\alpha_1}, \dots, T_{\alpha_{2g+1}}\}$.

Namely, Proposition 4.15 shows that $V = K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \cap W$ is spanned by the $\{\infty, P, Q\}$ where P and Q are in $\{O, O', T_{\alpha_1}, \dots, T_{\alpha_{2g+1}}\}$, and by Proposition 6.3 and its proof those elements can be expressed in the $\{\infty, O, T_{\alpha_i}\}$ or the $\{\frac{y^2}{x^d}, \frac{x-\alpha_i}{-\alpha_i}\}$. Therefore, if $k = \mathbb{Q}(\alpha_1, \dots, \alpha_{2g+1})$, then the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on V factors through $\mathrm{Gal}(k/\mathbb{Q})$, and $V^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is generated by the elements

$$\sum_{\sigma \in \mathrm{Gal}(k/\mathbb{Q})} \sigma\left(\left\{\frac{y^2}{x^d}, \frac{x-\alpha_i}{-\alpha_i}\right\}\right) = \sum_{\sigma \in \mathrm{Gal}(k/\mathbb{Q})} \left\{\frac{y^2}{x^d}, \frac{x-\sigma(\alpha_i)}{-\sigma(\alpha_i)}\right\} = \delta_i \left\{\frac{y^2}{x^d}, \frac{n_i(x)}{n_i(0)}\right\} ,$$

where $n_i(x)$ is the minimal polynomial of α_i over $\mathbb{Q}$ and $\delta_i \cdot \deg(n_i(x)) = [k : \mathbb{Q}]$. Because $n_i(x)$ is an irreducible rational factor of $t(x)$ this equals $\delta_i M_j$ for some j.

7. Constructing good polynomials

Our goal is to use Construction 6.11 to produce explicit families of hyperelliptic curves C with as many elements of $K_2^T(C)$ as possible. This comes down to constructing polynomials of the right form, which we will refer to as "good" polynomials. This is addressed in this section.

Problem 7.1. Construct (families of) polynomials of the form

$$t(x) = -x^d + f(x)^2/4$$

which have many rational factors and no multiple roots. Here $d \geq 5$, and we want that $t(x)$ has degree $d-1$ for d even, and degree d for d odd.

Remark 7.2. To get potentially interesting examples for the Beilinson conjecture, we need at least $g = \mathrm{genus}(C)$ linearly independent elements of $K_2^T(C)$/torsion. Thus we want $t(x)$ to have at least g rational factors if $d = 2g+1$ is odd and at least $g+1$ of them if $d = 2g+2$ is even (cf. (2) of Proposition 6.14). Such a $t(x)$ is what we will call a good polynomial.

We have seen in Examples 5.3 and 5.6 that, for our purposes, we might just as well consider, for $c \neq 0$,

$$t(x) = cx^{2g+1} + f(x)^2 \text{ where } f(x) = b_g x^g + \ldots + b_1 x + b_0$$

when $d = 2g+1$, and

$$t(x) = -c^2 x^{2g+2} + f(x)^2 \text{ where } f(x) = cx^{g+1} + b_g x^g + \ldots + b_1 x + b_0 \text{ with } b_g \neq 0$$

when $d = 2g+2$. It will be more convenient to use those non-normalized versions because we can sometimes let c play a role in the construction of such $t(x)$.

For the remainder of this section we keep the notations c, d, $t(x)$ and $f(x)$ as above. For $d = 5$ and $d = 6$ we will explain how to describe all such $t(x)$

that factor completely over the rationals. For larger degrees we will give sporadic examples that factor completely or nearly completely, and also infinite families of good polynomials (with $d = 2g + 2$) for all genera g.

One way to produce good polynomials is to start with a general polynomial $t(x)$ and force it to have given rational roots. We illustrate this with one example:

Example 7.3: $\mathbf{d = 5}$. Take

$$t(x) = x^5 + (b_2x^2 + b_1x + b_0)^2 \,.$$

and force the quintic to have two rational roots, so that we get (at least) 3 rational factors.

If α is a root of $t(x)$, then $-\alpha^5$ is a square, so $\alpha = -k^2$ for some $k \in \mathbb{Q}$. We want $t(x)$ to have two distinct rational roots $-m^2$ and $-n^2$, so

$$-m^{10} + (b_2m^4 - b_1m^2 + b_0)^2 = 0 \quad \text{and} \quad -n^{10} + (b_2n^4 - b_1n^2 + b_0)^2 = 0 \,.$$

Taking roots yields

$$m^5 = b_2m^4 - b_1m^2 + b_0 \quad \text{and} \quad n^5 = b_2n^4 - b_1n^2 + b_0 \,.$$

(We can take the positive signs by replacing m by $-m$ or n by $-n$ if necessary.) Solve this linear system for b_0 and b_1 and set $b_2 = (k + m^2 + mn + n^2)/(m + n)$. Finally, multiply by $(m + n)^2$ to obtain

$$(m+n)^2t(x) = (m+n)^2x^5 + \Big((k+m^2+mn+n^2)x^2 + (km^2+m^2n^2+kn^2)x + km^2n^2\Big)^2 \,.$$

This is a family on 3 parameters $m, n, k \in \mathbb{Q}$ although it gives only a 2-dimensional family of curves $y^2 = t(x)$: this equation is homogeneous of multi-degree $(1, 1, 2, 2, 5)$ in (m, n, k, x, y), so letting

$$m \mapsto \lambda m, \quad n \mapsto \lambda n, \quad k \mapsto \lambda^2 k \,,$$

gives a polynomial which corresponds to an isomorphic curve. Thus one can, for instance, assume that m, n, k are integers or instead that, say, $m = 1$. Note that $kmn \neq 0$ (for otherwise the quintic has a double root $\alpha = 0$) and that $m \neq \pm n$.

Remark 7.4. We could also have forced $t(x)$ to have a third rational root $\alpha = -l^2$. This condition gives a third linear equation for b_0, b_1 and b_2. Then the system can be solved uniquely, producing a 3-parameter family of polynomials with 4 rational factors which we do not write down here. We have been unable to find additional examples in this family for which we could verify Beilinson's conjecture, either because the coefficients were too large or because we did not obtain enough elements in $K_2(C; \mathbb{Z})$ of the corresponding curve C. Besides, there is a somewhat neater construction, which gives more rational roots:

Example 7.5: $\mathbf{d = 5}$. In degree 5 it is in fact possible to describe the polynomials of the desired form that factor completely over the rationals as follows. Recall from Example 7.3 that every rational root of $t(x)$ is of the form $-r^2$ for some rational r, so if $t(x)$ factors completely then we must have

$$t(x) = x^5 + (b_2x^2 + b_1x + b_0)^2 = \prod_{i=1}^{5}(x + r_i^2) \,.$$

Assume that the $|r_i|$ are pairwise distinct and non-zero (otherwise $t(x)$ has a double root). The two formulas for $t(x)$ give two factorizations of $-t(-x^2)$, and hence

$$(x^5 + b_2x^4 - b_1x^2 + b_0)(x^5 - b_2x^4 + b_1x^2 - b_0) \ = \ \prod_{i=1}^{5}(x - r_i)\prod_{i=1}^{5}(x + r_i)\,.$$

Call the two quintics on the left $q(x)$ and $-q(-x)$ respectively. Clearly in every pair $\pm r_i$ one of the numbers is a root of $q(x)$ and the other a root of $q(-x)$. By changing the signs of some of the r_i if necessary, we can assume that all of them are roots of $q(x)$. As the quintic $q(x)$ has no x^3 and x terms, Newton's formulae imply that the numbers $s_i = r_i^{-1}$ satisfy the two equations

$$\sum_{i=1}^{5} s_i = 0 \quad \text{and} \quad \sum_{i=1}^{5} s_i^3 = 0\,. \tag{7.6}$$

Conversely, any five-tuple $\{s_i\}$ of non-zero rationals satisfying (7.6) and such that $s_i \neq \pm s_j$ for $i \neq j$ gives rise to $t(x) = \prod(x + s_i^{-2})$ of the desired form. Note that the tuple $\{-s_i\}$ gives the same polynomial.

Now in order to find rational solutions of (7.6), note that these equations define a cubic surface in $\mathbb{P}^4$. So given two rational solutions to (7.6), the line joining them intersects the surface in a third point which is also rational. Thus, starting with some obvious solutions such as $(a, -a, b, -b, 0)$ and its permutations or some other experimentally found small solutions like

$$(1, 5, -7, -8, 9), \quad (2, 4, -7, -9, 10), \quad (1, 14, -17, -18, 20), \quad \dots \tag{7.7}$$

one can construct as many other solutions as one wants by using this "chord construction" and by scaling and permuting the coordinates.

Remark 7.8. The surface defined by (7.6) is the famous cubic surface of Clebsch and Klein, which also occurs as a Hilbert modular surface and was studied in detail in [10]. It might be worth investigating why this surface shows up in this context.

Example 7.9: d even. Finally, here is a method to produce examples for even $d = 2m$. First, if

$$t(x) = -c^2x^{2m} + f(x)^2/4 = (cx^m + f(x)/2)(f(x)/2 - cx^m) \tag{7.10}$$

with $f(x) = 2cx^m + b_gx^g + \dots + b_0$ and $cb_g \neq 0$, then we could take the first factor to be of the form $2c(x - a_i)\dots(x - a_m)$, which gives us at least $m+1$ rational factors of $t(x)$. In order to get more rational factors, it is easier to put $\tilde{f}(x) = x^m f(1/x)$, so that, with $g(x) = \tilde{f}(x)/2 + c$ and $k = 2c$,

$$x^{2m}t(1/x) = \tilde{f}(x)^2/4 - c^2 = g(x)(g(x) - k)\,, \tag{7.11}$$

and the problem simply becomes to find polynomials $g(x)$ and non-zero constants k such that $g(x)$ and $g(x) - k$ have no multiple roots and a lot of rational factors. Strictly speaking we should also ensure that (7.11) has no constant term and a non-zero linear term, so that (7.10) is of degree $2m - 1$, a condition which has to be satisfied to get the correct shape of $t(x)$. But this can be always achieved by a translation if there is a rational root.

Scaling $g(x)$ and k does not change our problem, so we can assume $g(x)$ has leading coefficient 1. If we choose $g(x) = (x - b_1)\dots(x - b_m)$ we get a family of good polynomials with at least $m + 1$ rational factors, having $b_1, \dots, b_m$ and k as

parameters. Now the question is how to improve this by forcing a polynomial of the form

$$g(x) - k = (x - b_1) \cdots (x - b_m) - k$$

to have more rational factors. If we choose $k = \prod(-b_i)$, then $g(x) - k$ has a factor x, so we get a family with at least $m+2$ rational factors, namely $x, x-b_1, \dots, x-b_m$ and the remaining factor of degree $m - 1$.

For $d = 6$ ($m = 3$) this remaining factor is quadratic. So with a suitable rational parametrization one can construct a "universal" family of polynomials which factor completely over the rationals. One example of such a family is given by

$$\begin{aligned} g(x) &= (x-1)(x-r-s)(x-rs), \\ k &= rs(r-1)(s-1), \\ g(x) - k &= (x-r)(x-s)(x-rs-1)\,. \end{aligned}$$

For larger degrees, examples can be found as follows. Without loss of generality we may assume that $b_1, \dots, b_m$ are integers. Choose bounds B, C and search through all integers $-B \le b_1 < \cdots < b_m \le B$. Let $g(x) = (x - b_1) \dots (x - b_m)$ and compute $g(-C), \dots, g(C-1), g(C)$. If a number $k \ne 0$ occurs more than once in this list of values, then $g(x) - k$ has several integral roots in the range from $-C$ to C, each of which yields one linear factor. This gives a simple method to look for polynomials (7.11) with even more rational roots.

Here are a few examples with $m=4$ and $m=5$ where both $g(x)$ and $g(x) - k$ factor completely, sorted according to k. The columns in the two tables below contain k, the roots of $g(x)$ and the roots of $g(x) - k$ respectively.

$m=4$		
k	$g(x)=0$	$g(x)=k$
2520	$9, 7, -4, -10$	$11, 2, 0, -11$
7560	$13, 12, -5, -14$	$16, 7, -2, -15$
10080	$14, 12, -4, -15$	$17, 6, 0, -16$
10080	$17, 14, -9, -17$	$19, 11, -7, -18$
12600	$17, 12, -6, -17$	$19, 8, -3, -18$
13860	$15, 12, -4, -17$	$18, 5, 1, -18$
15840	$15, 14, -3, -17$	$19, 5, 3, -18$
18720	$19, 12, -5, -19$	$21, 7, -1, -20$
25200	$19, 14, -3, -20$	$22, 5, 4, -21$
27720	$21, 12, -5, -22$	$23, 6, 0, -23$

$m=5$		
k	$g(x)=0$	$g(x)=k$
50400	$12, 11, -1, -6, -14$	$14, 6, 4, -9, -13$
50400	$14, 10, 1, -9, -15$	$15, 6, 5, -11, -14$
110880	$15, 13, -3, -4, -16$	$17, 8, 4, -9, -15$
110880	$16, 10, -3, -7, -16$	$17, 5, 4, -12, -14$
272160	$17, 16, -1, -7, -20$	$20, 8, 7, -11, -19$
327600	$20, 11, -5, -9, -21$	$21, 5, 4, -15, -19$
393120	$23, 17, -6, -15, -24$	$24, 15, -3, -19, -22$
554400	$21, 19, 2, -11, -24$	$24, 11, 9, -14, -23$
554400	$22, 17, -1, -9, -24$	$24, 11, 6, -13, -23$
1058400	$23, 22, -1, -8, -27$	$27, 13, 8, -13, -26$

One should note here that such a search produces many more examples than just the ones above. The tables actually start

$m=4$		
k	$g(x)=0$	$g(x)=k$
180	$5, 4, -3, -4$	$6, 2, -1, -5$
360	$6, 4, -3, -5$	$7, 1, 0, -6$
504	$7, 5, -4, -6$	$8, 3, -2, -7$
720	$7, 4, -4, -7$	$8, 1, -1, -8$
...	...	...

$m=5$		
k	$g(x)=0$	$g(x)=k$
5040	$8, 7, -1, -5, -9$	$9, 5, 1, -7, -8$
10080	$9, 7, -2, -4, -10$	$10, 4, 2, -7, -9$
50400	$12, 11, -1, -6, -14$	$14, 6, 4, -9, -13$
50400	$14, 10, 1, -9, -15$	$15, 6, 5, -11, -14$
...	...	...

However, most of the examples are induced in the sense that modulo a linear substitution, $p(x) = g(x)(g(x) - k)$ is of the form $q(x^2)$ for some $q(x)$ of degree m. For instance, for the first entry for $m=4$ ($k=180$) we have

$$p(x) = (x-5)(x-4)(x+3)(x+4) \cdot (x-6)(x-2)(x+1)(x+5)$$

and, after rearranging the factors, we find that

$$\begin{aligned} p(-x+\tfrac{1}{2}) &= (x-\tfrac{11}{2})(x-\tfrac{9}{2})(x-\tfrac{7}{2})(x-\tfrac{3}{2})(x+\tfrac{3}{2})(x+\tfrac{7}{2})(x+\tfrac{9}{2})(x+\tfrac{11}{2}) \\ &= x^8 - 65x^6 + \tfrac{10979}{8}x^4 - \tfrac{164385}{16}x^2 + \tfrac{4322241}{256}\,. \end{aligned}$$

The curve $y^2 = t(x)$ is therefore a cover of an elliptic curve $y^2 = x^4 - 65x^3 + \cdots$. In the same way, the first two lines for $m=5$ simply come from genus 2 curves, which we have already seen in the cubic surface construction: they are given by the first two 5-tuples in (7.7).

Finally, for $m = 6$ ($g=5, d=12$) one can easily find several non-trivial examples where $g(x)(g(x)-k)$ factors completely except for one quadratic factor. It is harder to find non-trivial examples which factor completely, but they do exist. In the notation of (7.11) the smallest one is

$$\begin{aligned} &(x^6+2x^5-787x^4-188x^3+150012x^2-149040x-3326400)^2 \;-\; 3326400^2 \\ &= (x-22)(x-20)(x-18)(x-12)(x-10)(x-1)x(x+7)(x+15)(x+18)(x+23)(x+24)\,. \end{aligned}$$

8. Integrality of the elements

Construction 6.11 shows how to construct hyperelliptic curves $C/\mathbb{Q}$ with elements in $K_2^T(C)/\text{torsion}$. However, we are not yet in a position to compute regulators and test Beilinson's conjecture 3.11, because we still need our elements to be *integral*, that is, to lie in $K_2(C;\mathbb{Z}) \subseteq K_2^T(C)/\text{torsion}$. (See (3.7).)

Let $\mathcal{C}/\mathbb{Z}$ be a regular proper model of $C/\mathbb{Q}$. Recall that an element α of $K_2(\mathbb{Q}(C))$ lies in $K_2^T(C)$ if the tame symbol $T_P(\alpha)$ is trivial for all $P \in C(\overline{\mathbb{Q}})$. Recall also that α lies in $K_2^T(\mathcal{C})$ if the tame symbol is trivial for each irreducible curve $\mathcal{D}$ in $\mathcal{C}$; see (3.5) or (3.9). This means that apart from being trivial for "horizontal" curves (which come from points $P \in C(\overline{\mathbb{Q}})$), the tame symbol of α must be trivial for all irreducible components of the fibers $\mathcal{C}_p$ of $\mathcal{C} \to \operatorname{Spec}\mathbb{Z}$. This gives additional conditions on α for all primes p of $\mathbb{Z}$. (One can show though that, to a given α, one can always add an element in $K_2(\mathbb{Q})$ such that the sum satisfies this condition for each prime p for which the fiber $\mathcal{C}_p$ is smooth over $\mathbb{F}_p$.)

So, in general, given a curve $C/\mathbb{Q}$ and $\alpha \in K_2^T(C)$ the way to verify that α gives rise to an element of $K_2(C;\mathbb{Z})$ is to find a regular model $\mathcal{C}/\mathbb{Z}$ and then check that the tame symbol of α at each "vertical" curve on $\mathcal{C}$ is trivial. In practice, finding such a model means starting with any equation of C with integer coefficients, which defines an arithmetic surface, and performing blow-ups until we obtain a regular surface. This, however, is a complicated process, so we will try to deduce integrality from the behaviour of α on the original (possibly singular) arithmetic surface.

This can be done in fair generality for the families of examples that we are interested in. We repeat our notation for the sake of easy reference, so we consider a curve of genus g as in Construction 6.11, defined by

$$y^2 + f(x)y + x^d = 0 \tag{8.1}$$

in one of the two cases

$$\text{(8.2)}\quad \begin{array}{lll} \text{(a)} & d = 2g+1, & f(x) = b_g x^g + \ldots + b_1 x + b_0\,; \\ \text{(b)} & d = 2g+2, & f(x) = 2x^{g+1} + b_g x^g + \ldots + b_1 x + b_0 \text{ and } b_g \neq 0\,, \end{array}$$

where $t(x) = -x^d + f(x)^2/4$ has no multiple roots, so that in particular $b_0 \neq 0$. Assume further that $f(x)$ has integer coefficients, so (8.1) defines an arithmetic surface over $\mathbb{Z}$.

Theorem 8.3. *Let $C/\mathbb{Q}$ be defined by (8.1) and (8.2), where $b_0, \ldots, b_g$ are integers. Let $m(x) \in \mathbb{Z}[x]$ be a non-constant factor, irreducible over $\mathbb{Z}$, of the 2-torsion polynomial $t(x) = -x^d + f(x)^2/4$ and let M be the class of*

$$\left\{ \frac{y^2}{x^d}, \frac{m(x)}{m(0)} \right\},$$

in $K_2^T(C)/\text{torsion}$. We have

(1) *if $m(0) = \pm 1$ then $M \in K_2(C;\mathbb{Z})$;*
(2) *if there is a prime dividing $m(0)$ but not every b_i then $nM \notin K_2(C;\mathbb{Z})$ for any $n \neq 0$.*

Moreover,

(3) *if $b_0 = \pm 1$ and $\mathbb{M}$ is as in Proposition 6.3 then $\mathbb{M}$ is in $K_2(C;\mathbb{Z})$ if $d = 2g+1$ and $2\mathbb{M}$ is in $K_2(C;\mathbb{Z})$ if $d = 2g+2$.*

Proof. Denote

$$\widetilde{M} = \{f_1, f_2\}, \qquad f_1 = \frac{y^2}{x^d}, \quad f_2 = \frac{m(x)}{m(0)};$$

then $\widetilde{M}$ is an element of $K_2^T(C)$ (see (6.10)) and M is its class in $K_2^T(C)/\text{torsion}$. We shall show that, for a specific regular proper model $\mathcal{C}$ of C, $\widetilde{M} \in K_2^T(\mathcal{C})$ if $m(0) = \pm 1$, so that $M \in K_2(C;\mathbb{Z})$. But, for the same model, we shall show that under the asssumptions of (2), $T_{\mathcal{D}}(\widetilde{M})$ is not torsion for some irreducible component $\mathcal{D}$ of the fiber $\mathcal{C}_p$. Therefore no non-zero multiple of M can lie in $K_2(C;\mathbb{Z}) \subseteq K_2^T(C)/\text{torsion}$, because, up to torsion, $K_2^T(\mathcal{C})$ is a subgroup of $K_2^T(C)$ that does not depend on the choice of $\mathcal{C}$. Finally, for (3) we show that $\{-b_0 y, -x\}$ or $2\{-b_0 y, -x\}$ is in $K_2^T(\mathcal{C})$.

Before we can give the proof proper, we need some preliminaries.

Let p be a prime and F the fiber above p of the arithmetic surface defined by (8.1). It is clear from the equation that any irreducible component D of F dominates the x-axis and that x, y and $m(x)$ define rational functions on D that do not vanish identically on it.

We want to know that if p does not divide every b_i then the function y^2/x^d is non-constant on every such D. If it were constant then $y^2 = kx^d$ on D for some constant $k \neq 0$. So $f(x)y = -(k+1)x^d$ on D and squaring this equation yields $f(x)^2 k x^d = (k+1)^2 x^{2d}$. Since D dominates the x-axis, this is only possible if this is an identity of polynomials, which implies that $b_g x^g + \cdots + b_0 = 0 \mod p$, contradicting the assumption on the b_i.

Let $\mathcal{C}/\mathbb{Z}$ be the regular model of $C/\mathbb{Q}$ obtained from the arithmetic surface defined by (8.1) by repeatedly blowing up the singularities. If $\mathcal{D} \subset \mathcal{C}$ is an irreducible curve mapping onto D, then the function $f_1 = \frac{y^2}{x^d}$ is non-constant along $\mathcal{D}$ as well.

We are now ready to prove (2). Let p divide $m(0)$ but not every b_i. Let D and $\mathcal{D}$ be as above, so that $v_{\mathcal{D}}(f_1) = 0$ and $v_{\mathcal{D}}(f_2) < 0$. Then the tame symbol $T_{\mathcal{D}}(\widetilde{M})$ is a non-zero power of f_1 and is therefore non-constant and hence non-torsion.

We now move to the proof of (1) so we assume that $m(0) = \pm 1$. Then for any prime p and irreducible component $\mathcal{D}$ of $\mathcal{C}_p$ surjecting onto a component D of the fiber above p of (8.1), $v_{\mathcal{D}}(f_1) = v_{\mathcal{D}}(f_2) = 0$, hence $T_{\mathcal{D}}(\widetilde{M}) = 1$.

However, this does not prove that $v_{\mathcal{D}}(f_1) = v_{\mathcal{D}}(f_2) = 0$ for *every* component $\mathcal{D}$ of the fiber of $\mathcal{C} \to \operatorname{Spec}\mathbb{Z}$ above p: such a $\mathcal{D}$ could also map to a singular point of

the model (8.1). In this case if, say, f_1 happens to have a zero or a pole passing just through this point, then it can happen that $v_{\mathcal{D}}(f_1) \neq 0$.

We claim that at every singular point P of the surface defined by (8.1) either f_1 or f_2 is regular and equal to 1. This implies that $T_{\mathcal{D}}(\widetilde{M}) = 1$ for every irreducible curve $\mathcal{D}$ of $\mathcal{C}$ mapping to P, proving (1).

In fact, our claim holds for all singular points in the fiber F of (8.1) above a prime p, among which are those singularities of $\mathcal{C}$ that lie above p. In order to see this, first consider the point at infinity in F. The arithmetic surface defined by (8.1) has a chart at infinity which can be obtained by letting $x = 1/\tilde{x}$ and $y = \tilde{y}/\tilde{x}^{g+1}$. So for $d = 2g+1$, the equation at infinity is

$$\tilde{y}^2 + (b_g\tilde{x} + \ldots + b_1\tilde{x}^g + b_0\tilde{x}^{g+1})\,\tilde{y} + \tilde{x} = 0$$

and the point at infinity in F ($\tilde{x} = \tilde{y} = 0 \bmod p$) is non-singular. If $d = 2g+2$, then the equation is

$$\tilde{y}^2 + (2 + b_g\tilde{x} + \ldots + b_1\tilde{x}^g + b_0\tilde{x}^{g+1})\,\tilde{y} + 1 = 0$$

and the function $f_1 = \tilde{y}^2$ is regular and equal to 1 at the point at infinity, $(\tilde{x}, \tilde{y}) = (0, -1) \bmod p$.

For finite x, if $P = (x_0, y_0) \mod p$ is a singularity of F then $f(x_0) = -2y_0$. Then either $x_0 \neq 0$, in which case f_1 is regular and equal to 1 at P from the equation of the curve, or $x_0 = 0$, in which case f_2 is regular and equal to 1 at P because p does not divide $m(0)$. This completes the proof of (1).

For (3), assume that $b_0 = \pm 1$, so that $\mathbb{M}$ is the class in $K_2^T(C)/\text{torsion}$ of the element $\{-b_0 y, -x\}$ in $K_2^T(C)$. As above, there can only be a problem at a singular point of the arithmetic surface defined by (8.1), and that point is a singularity on the fiber F above some prime p. But if $P = (x_0, y_0)$ modulo p is any finite singularity of F, then $2y_0 + f(x_0) = 0$, hence $y_0^2 = x_0^d$. We cannot have $(x_0, y_0) = (0, 0)$ since this would give $\pm 1 = b_0 = f(x_0) = -2y_0 = 0$. Therefore neither $-b_0 y = \mp y$ nor $-x$ can have a zero or a pole passing through P, showing that those functions have trivial valuation along any irreducible curve $\mathcal{D}$ in $\mathcal{C}$ mapping to P. For $d = 2g+1$ we saw before that the point at infinity of F is non-singular, so that $\mathbb{M}$ is in $K_2(C;\mathbb{Z})$. For $d = 2g+2$ we consider instead the class $2\mathbb{M}$ of $2\{-b_0 y, -x\} = \{y^2, x\} = \{y^2/x^{2g+2}, x\} = \{\tilde{y}^2, 1/\tilde{x}\}$, and at the point at infinity of F, $(\tilde{x}, \tilde{y}) = (0, -1)$ modulo p, $\tilde{y}^2$ is regular and equal to 1. □

Remark 8.4. We cannot drop the assumption in (2) of Theorem 8.3 that the prime does not divide each b_i, i.e., the unrestricted converse of (1) does not hold. In fact, given $m(x)$ as in the theorem with $m(0) = \pm 1$, so that the corresponding M is in $K_2(C;\mathbb{Z})$, fix a prime p and a positive integer s. Then the substitution $(x, y) \mapsto (p^{-2s}x, p^{-ds}y)$. transforms C into an isomorphic curve $\widetilde{C}$ defined by

$$y^2 + \tilde{f}(x)y + x^d = 0\,,$$

where $\tilde{f}(x) = p^{ds} f(p^{-2s}x)$ so that all $\tilde{b}_i$ are divisible by p. Choosing $k \geq 0$ minimally such that $\widetilde{m}(x) = p^k m(p^{-2s}x)$ is in $\mathbb{Z}[x]$, $\widetilde{m}(x)$ is irreducible in $\mathbb{Z}[x]$ and is a non-constant factor of the 2-torsion polynomial $-x^d + \tilde{f}(x)^2/4 = p^{2ds}t(p^{-2s}x)$. We can ensure that $k > 0$ by choosing s sufficiently large, so that p divides $\widetilde{m}(0)$. Then the class of $\{y^2/x^d, \widetilde{m}(x)/\widetilde{m}(0)\}$ is in $K_2(\widetilde{C};\mathbb{Z})$ since it corresponds to M, but p divides $\widetilde{m}(0)$ as well as each $\tilde{b}_i$.

9. Computing the Beilinson regulator

Let C be a hyperelliptic curve of genus g over $\mathbb{C}$, so the map $\phi : C \to \mathbb{P}^1_{\mathbb{C}}$ has $n = 2g+2$ points of ramification, $P_1, \ldots, P_n$, and let $X = C(\mathbb{C})$ be the associated Riemann surface. It is not difficult to check that a basis of $H_1(X;\mathbb{Z})$ consists of liftings to X of simple loops in $\mathbb{P}^1_{\mathbb{C}}$ around exactly P_i and P_{i+1}, for $i = 1, \ldots, 2g$. If a model of C is defined by an equation $y^2 = f(x)$ with $f(x)$ in $\mathbb{C}[x]$ of degree $2g+1$ without multiple roots, the point in C above the point at infinity in this model will be among the ramification points, and the other ones will be the points above the roots of $f(x)$ in $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$. We therefore work with simple loops in $\mathbb{C}$ around exactly two roots of $f(x)$, not passing through any root of $f(x)$, and lift them to loops on X.

In our case $f(x)$ belongs to $\mathbb{Q}[x]$ and hence to $\mathbb{R}[x]$, and we want to keep track of the action of complex conjugation on X. We will discuss the case when the leading coefficient of $f(x)$ is positive, which we can always achieve by replacing x with $-x$ if necessary. The set $C(\mathbb{R})$ of real points on X consists of ∞_C together with the points above those x in $\mathbb{R} \subset \mathbb{C}$ where $f(x) \geq 0$. In Figure 9.1, the latter is indicated by the thick part of $\mathbb{R}$ in $\mathbb{C}$, the thick dots being the roots of $f(x)$.

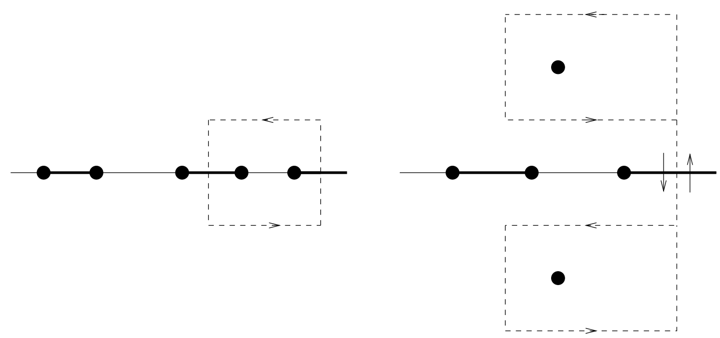

Figure 9.1.

If $f(x)$ has only real roots $P_1 < P_2 < \cdots < P_{2g+1}$, then let γ_i be a lift of a loop around P_{2i} and P_{2i+1} ($i = 1, \ldots, g$) (this is illustrated on the left in Figure 9.1), and let δ_i be a lift of a loop around P_{2i-1} and P_{2i} ($i = 1, \ldots, g$). Since all of the γ_i have a fixed point under the action of complex conjugation, which lies above the intersection with the thick part of the real line, it is easy to check that the γ_i lie in $H_1(X;\mathbb{Z})^-$ because complex conjugation reverses the orientation. Also, by shrinking the δ_i as much as possible, so that they lift to a subset of $C(\mathbb{R})$, it is easy to see that the δ_i lie in $H_1(X;\mathbb{Z})^+$. So in this case $H_1(X;\mathbb{Z})$ decomposes as a direct sum $H_1(X;\mathbb{Z})^- \oplus H_1(X;\mathbb{Z})^+$, the γ_i form a basis of $H_1(X;\mathbb{Z})^-$, and the δ_i form a basis of $H_1(X;\mathbb{Z})^+$.

If $f(x)$ does not have only real roots, let $P_1, \ldots, P_{2m+1}$ be the real roots of $f(x)$, and let $Q_1, \overline{Q_1}, \ldots, Q_{g-m}, \overline{Q_{g-m}}$ be its non-real roots, as pairs of complex conjugated numbers with $\mathrm{Im}(Q_j) > 0$. We now define $\gamma_1, \ldots, \gamma_m$ as lifts of simple loops around P_2 and P_3, P_4 and P_5, $\ldots$, P_{2m} and P_{2m+1}, and $\gamma_{m+1}, \ldots, \gamma_g$ as lifts of simple loops around Q_1 and $\overline{Q_1}$, Q_2 and $\overline{Q_2}$, etc., intersecting the thick part of the real axis as illustrated on the right in Figure 9.1. Then it is easy to check that $\gamma_1, \ldots, \gamma_g$ are in $H_1(X;\mathbb{Z})^-$. Also, if $\delta_1, \ldots, \delta_g$ are lifts to X of simple loops around

P_1 and P_2, …, P_{2m-1} and P_{2m}, as well as around P_{2m+1} and Q_1, Q_1 and Q_2, Q_2 and Q_3, etc., then the δ_j complement the γ_j to a basis of $H_1(X;\mathbb{Z})$. Note that in Figure 9.2 we still have $\delta' \circ \sigma = \delta'$ as in the case when $f(x)$ has only real roots, but

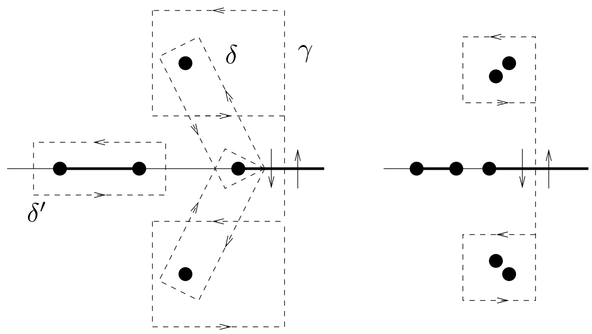

FIGURE 9.2.

we do not get a splitting of $H_1(X;\mathbb{Z})$ into $+$ and $-$-parts. Namely, if δ is the lift of a loop around P_{2m+1} and Q_1, and γ is the lift of a loop around Q_1 and $\overline{Q_1}$ passing through the thick part of the real line to the right of P_{2m+1}, then $\delta - \delta \circ \sigma = \gamma$, with γ in $H_1(X;\mathbb{Z})^-$, provided δ and γ are chosen in such a way that they are in the same branch of $C(\mathbb{R})$ above the thick part of the real line through which both loops in $\mathbb{C}$ pass. (An illustration of this is given on the right in the left part of Figure 9.2.) This holds because looping twice around a single point lifts to the trivial element in $H_1(X;\mathbb{Z})$. If we could write $\delta = \delta^+ + \delta^-$ with $\delta^\pm$ in $H_1(X;\mathbb{Z})^\pm$, then this would lead to $2\delta^- = \delta - \delta \circ \sigma = \gamma$, which is not possible since γ is part of a basis of $H_1(X;\mathbb{Z})$. In summary we find in this case that $\{\gamma_1, \dots, \gamma_g\}$ is a basis of $H_1(X;\mathbb{Z})^-$, and that $\{\delta_1, \dots, \delta_g\}$ is a basis of $H_1(X;\mathbb{Z})/H_1(X;\mathbb{Z})^-$.

For practical purposes, we choose all our loops in $\mathbb{C}$ as concatenations of line segments, which makes computing and parametrizing their lifts to X using the analytic continuation of square roots particularly easy. Apart from the two types already described (lifts of a simple loop around two consecutive real roots of $f(x)$, or of a simple loop around two conjugate non-real roots of $f(x)$ intersecting the thick part of the real line, both illustrated in Figure 9.1), we also use a third type in order to avoid getting close to roots of $f(x)$ unnecessarily, thus speeding up the calculations in numerical integration. If, for example, $Q_1 = x_1 + iy_1$ and $Q_2 = x_2 + iy_2$ are two non-real roots of $f(x)$ which are close together, and $0 < y_1 \le y_2$, then, keeping a lift of a loop around the pair $Q_1, \overline{Q_1}$ meeting the thick part of the real line, we can replace a lift of a loop around the pair $Q_2, \overline{Q_2}$ with a loop around both pairs $Q_1, \overline{Q_1}, Q_2, \overline{Q_2}$, still meeting the thick part of the real line, as indicated on the right in Figure 9.2. This is just the sum of (compatible) lifts of a loop around $Q_1, \overline{Q_1}$ and a loop around $Q_2, \overline{Q_2}$, both meeting the thick part of the real line in a common point. A similar method is used if more than two non-real roots of $f(x)$ are close together.

Finally, we choose our loops also such that they do not pass through the image of the other torsion point under the map $\phi : X \to \mathbb{P}^1_{\mathbb{C}}$. This way, we can immediately compute $\langle \gamma, \{f, g\} \rangle = \frac{1}{2\pi} \int_\gamma \eta(f,g)$, with $\eta(f,g)$ as in (3.3), and integrate numerically over the lifts of the paths using the analytic continuation of $y = \sqrt{f(x)}$

along the paths. The method works quite well in practice and we can easily obtain twelve or fifteen decimals of precision for the regulator.

10. Examples

In this section we test Beilinson's conjecture 3.11 as explained in Remark 3.13. We do this for hyperelliptic curves over $\mathbb{Q}$, of genus 2, 3, 4 and 5. Using the results of Section 7, we construct curves of the form (8.1) and (8.2) whose 2-torsion polynomials have many rational factors. As Construction 6.11 shows, every such rational factor gives an element of $K_2^T(C)/\text{torsion}$. Theorem 8.3 shows which of these are in $K_2(C;\mathbb{Z})$.

Our main sources of examples are the constructions described in Examples 7.3 and 7.9 (or variations of them). Unfortunately, the more sophisticated constructions of Examples 7.4 and 7.5 tend to produce either not enough symbols satisfying the integrality condition or else curves of very high conductor for which we cannot compute the L-value.

Example 10.1: genus 2. Taking $m = 1$ and $n = -2$ in Example 7.3 yields a one-parameter family of curves

$$y^2 = x^5 + \Big((k+3)x^2 + (5k+4)x + 4k\Big)^2 .$$

Further, if k is an integer which is divisible by 4, then the curve can be transformed to one of the form in Theorem 8.3. Thus write $k=-4b$ with $b \in \mathbb{Z}$. The curve is then isomorphic to

$$C_b : \quad y^2 + ((4b-3)x^2 - (5b-1)x + b)y + x^5 = 0 .$$

For non-zero b this is a non-singular genus 2 curve of discriminant

$$\Delta(C_b) = 3^2 b^7 (8b+3)^2 (9b^3 - 23b^2 + 4b - 1) .$$

Its 2-torsion polynomial has 3 rational factors, namely (up to a constant)

$$\begin{aligned} m_1 &= x-1, \\ m_2 &= 4x-1, \\ m_3 &= x^3 - (4b^2-6b+1)x^2 + (5b^2-b)x - b^2 . \end{aligned}$$

Recall from Construction 6.11 that each m_i gives an element $M_i \in K_2^T(C)/\text{torsion}$. Intersecting the lattice in $K_2^T(C)/\text{torsion}$ which they span with $K_2(C;\mathbb{Z})$ we obtain a sublattice of integral elements,

$$\Lambda_M = \langle M_1, M_2, M_3 \rangle \ \cap \ K_2(C;\mathbb{Z}).$$

Using Theorem 8.3 we can determine Λ_M exactly by inspecting the constant terms of the m_i. We find that $\Lambda_M = \langle M_1, M_2, M_3\rangle$ for $b = \pm 1$ and $\Lambda_M = \langle M_1, M_2\rangle$ otherwise.

We can now test Beilinson's conjecture for this family. We constructed at least two elements (M_1 and M_2) of Λ_M and we might *generally* expect them to be linearly independent. If this is the case, Λ_M should be a subgroup of finite index in $K_2(C;\mathbb{Z})$, which is supposed to have rank 2 by the first part of Beilinson's conjecture 3.11. The second part would then imply that $R(\Lambda_M)$, the regulator computed using a basis of Λ_M, is a non-zero rational multiple of $L^*(C_b, 0)$, the leading coefficient of $L(C_b, s)$ at $s=0$. In summary, we expect

(a) $\mathrm{rk}_{\mathbb{Z}} \Lambda_M = 2$;

(b) $R(\Lambda_M) = QL^*(C_b, 0)$ for some $Q \in \mathbb{Q}^*$.

Note that the prediction $\operatorname{rk}\Lambda_M = 2$ implies that in the case $b = \pm 1$, where we have $\Lambda_M = \langle M_1, M_2, M_3\rangle$, there should be a linear relation among the M_i. Let us illustrate the latter point in the case $b = -1$. We can compute the images of the M_i under the regulator pairing (3.10), as explained in Section 9,

$$\alpha \longmapsto (\langle\gamma_1, \alpha\rangle, \langle\gamma_2, \alpha\rangle) \in \mathbb{R}^2, \tag{10.2}$$

where γ_1 and γ_2 form a basis of $H_1(X;\mathbb{Z})^-$. Since we expect the image to be a lattice of rank 2, there should be some integral linear combination of the M_i for which the image in $\mathbb{R}^2$ vanishes. Using LLL to look for a small $\mathbb{Z}$-relation between the images in $\mathbb{R}^2$, we find that

$$41M_1 + 56M_2 - 44M_3 \overset{?}{\longmapsto} (0,0)\,. \tag{10.3}$$

To make the relation more transparent we complete the vector $(41, 56, -44)$ to a unimodular integral matrix and make the corresponding change of basis:

$$\begin{pmatrix} M_1^* \\ M_2^* \\ M_3^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 11 & 15 & 0 \\ 41 & 56 & -44 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}.$$

The numerical values of the regulator pairing (10.2) on the M_i and the M_i^* are then given by the table

α	M_1	M_2	$M_3 = M_1^*$	M_2^*	M_3^*
$\langle\gamma_1, \alpha\rangle$	-0.519837	-6.055409	-8.191279	-96.549352	0.000000
$\langle\gamma_2, \alpha\rangle$	3.243279	-0.869701	1.915254	22.630553	0.000000

This strongly suggests that M_3^* is in the kernel of (10.2), in which case M_3^* should be zero in Λ_M by Beilinson's conjecture. The regulator $R(\Lambda_M)$ is now obtained as the absolute value of the determinant of the 2×2 matrix which consists of the two columns for M_1^* and M_2^*,

$$R(\Lambda_M) = \begin{vmatrix} -8.191279 & -96.549352 \\ 1.915254 & 22.630553 \end{vmatrix} = 0.456625\,.$$

Numerically, $L^*(C_{-1}, 0) \approx 0.228312$, so $L^*(C_{-1}, 0)/R(\Lambda_M) \approx 0.500000$, which we "recognize" as the rational number $1/2$. (Here the agreement is to at least 18 digits, which was our working precision in this example.)

Recall from Theorem 8.3 that for $b = \pm 1$ the class $\mathbb{M}$ of $\{-by, -x\}$ also lies in $K_2(C;\mathbb{Z})$ and satisfies $10\mathbb{M} = M_1 + M_2 + M_3$ by (1) of Proposition 6.14. For $b = -1$, we can write this as $10\mathbb{M} = -175M_1^* + 15M_2^* - 4M_3^*$, so if we assume $M_3^* = 0$, then $2\mathbb{M} = -35M_1^* + 3M_2^*$ is in Λ_M but $\mathbb{M}$ is not. In other words, the larger lattice $\Lambda = \langle M_1, M_2, M_3, \mathbb{M}\,\rangle$ in $K_2(C;\mathbb{Z})$ contains Λ_M as a sublattice of index 2 and

$$L^*(C_{-1}, 0)/R(\Lambda) = 1.000000... \overset{?}{=} 1\,.$$

Table 1 summarizes the computations for the curves C_b with $|b| \le 10$. In the third column we describe the lattice Λ that we use, and the last column contains the quotient of $L^*(C_b, 0)$ and the Beilinson regulator $R(\Lambda)$. In all cases this quotient appears to be a rational number of relatively small height. For $b = \pm 1$ we also include the expected relation between the M_i. (We have not included the universal relation $10\mathbb{M} = M_1 + M_2 + M_3$ because similar relations would clog up later tables.)

Finally, we remark that the Jacobian of each of the curves appearing in Table 1, except for $b = 1$, is not isogenous to the product of two elliptic curves, even over $\overline{\mathbb{Q}}$. This was verified using the procedure of [6, Chapter 14, Section 4]. In other words, the elements in K-theory and the L-values do not come from elliptic curves, and the verification of Beilinson's conjecture does not reduce to the case of genus 1. The same holds for all the curves in Table 2 (Example 10.5) and the curves in Table 6 (Example 10.8) except the one with $v = \{2, 5\}$.

Notation 10.4. In all of the following examples we keep the same notation as above. For a curve

$$C : y^2 + f(x)y + x^d = 0$$

as in (8.1) and (8.2), denote by m_i the rational factors of the 2-torsion polynomial $t(x) = -x^d + f(x)^2/4$. The associated elements of $K_2^T(C)/\text{torsion}$ (see (6.12)) are denoted by M_i. Using $\mathbb{M}$ or $2\mathbb{M}$ to aim for as large a subgroup of $K_2(C;\mathbb{Z})$ as possible (see Theorem 8.3) we let

$$\Lambda' = \begin{cases} \langle M_1, \dots, M_k \rangle, & f(0) \neq \pm 1, \\ \langle M_1, \dots, M_k, \mathbb{M} \rangle, & f(0) = \pm 1, d = 2g+1, \\ \langle M_1, \dots, M_k, 2\mathbb{M} \rangle, & f(0) = \pm 1, d = 2g+2, \end{cases}$$

and use the sublattice $\Lambda = \Lambda' \cap K_2(C;\mathbb{Z})$ of integral elements. In all of the examples below we can determine Λ by applying Theorem 8.3. Then we verify numerically that Λ has rank equal to the genus of C and that the leading coefficient $L^*(C, 0)$ is a rational multiple of the regulator $R(\Lambda)$ of Λ. To keep the entries for Λ simple, we did not include the universal relation between $\mathbb{M}$ and the M_i of (1) of Proposition 6.14 in the tables.

Example 10.5: genus 2. We can construct another family of genus 2 curves with enough elements in $K_2(C;\mathbb{Z})$ in a way similar to Example 7.9, this time using 6-torsion points. The curves are given by

$$y^2 + (2x^3 - 4bx^2 - x + b)y + x^6 = 0\,, \qquad b \in \mathbb{Z}_{>0}.$$

The 2-torsion polynomial has 4 rational factors,

$$m_1 = x - b, \quad m_2 = 2x - 1, \quad m_3 = 2x + 1, \quad m_4 = 4bx^2 + x - b,$$

so we get 4 elements of $K_2^T(C)/\text{torsion}$, again denoted by M_1, M_2, M_3, M_4. According to (2) of Proposition 6.14 these satisfy the relation $M_1 + M_2 + M_3 = M_4$ so M_4 can be dropped. Furthermore, M_2 and M_3 are always integral, and M_1 is for $b=1$. The numerical results, and the expected relation between the M_i when $b=1$, are given in Table 2.

Example 10.6: genus 3. We can look at hyperelliptic genus 3 curves with a rational 7-torsion point and whose 2-torsion polynomial has two given rational root, cf. Example 7.3. A special case of this is the two-parameter family given by

$$y^2 + ((4b-3)x^3 - (4a+5b-1)x^2 + (5a+b)x - a)y + x^7 = 0\,.$$

The curves have a rational 7-torsion point $(0, 0)$ and two rational 2-torsion points with x-coordinates 1 and 1/4. The 2-torsion polynomial has 3 rational factors,

$$\begin{aligned} m_1 &= x - 1, \\ m_2 &= 4x - 1, \\ m_3 &= x^5 - (4b^2-6b+1)x^4 + (5b^2+8ab-2b-6a)x^3 \\ &\qquad\qquad -(b^2+10ab+4a^2-2a)x^2 + (5a^2+2ab)x - a^2\,. \end{aligned}$$

For $a = \pm 1$ and $b \in \mathbb{Z}$ we thus get 3 integral symbols and we expect a relation with $L^*(C, 0)$. Moreover, for $a=1, b=3$ we have a further factorization

$$m_3 = m_{3,1} m_{3,2} = (x^2-3x+1)(x^3-16x^2+8x-1).$$

so we get 4 elements of $K_2(C;\mathbb{Z})$ and we expect them to be linearly dependent. The numerical results, and the expected relation for $a = 1$, $b = 3$, are summarized in Table 3.

Example 10.7: genus 4, 5. We can also give some sporadic examples of curves of genus $g=4$ and $g=5$, given by

$$y^2 + f(x)y + x^{10} = 0$$

with $f(x)$ of degree 5 and

$$y^2 + f(x)y + x^{12} = 0$$

with $f(x)$ of degree 6 respectively. These can be found by a brute search for good polynomials with enough factors in $\mathbb{Z}[x]$ with constant term ± 1. In all cases that we found, we have exactly $g + 1$ such factors and the lattice $\Lambda = \langle M_1, \ldots, M_g, 2\mathbb{M}\rangle$ is of rank g. The examples and the corresponding numerical results are given in Tables 4 and 5.

Example 10.8: arbitrary genus. Finally, we give examples of curves of arbitrary genus g with g integral symbols. Consider a curve of the form

$$y^2 + \Big(2x^{g+1} \pm \prod_{j=1}^{g}(v_j x + 1)\Big)\, y + x^{2g+2} = 0\,, \tag{10.9}$$

where $v_1 < \cdots < v_g$ are distinct non-zero integers, and we are assuming that the 2-torsion polynomial $t(x)$ has no multiple roots. Then $t(x)$ has at least $g+1$ factors in $\mathbb{Z}[x]$,

$$v_1 x + 1,\ v_2 x + 1, \ldots, v_g x + 1,\ 4x^{g+1} \pm \prod_{j=1}^{g}(v_j x + 1)\,. \tag{10.10}$$

(This is similar to, but different from, the discussion in Example 7.9 because now, in $4t(x) = -4x^{2g+2} + f(x)^2 = (f(x) - 2x^{g+1})(2x^{g+1} + f(x))$, we decompose the first factor, which has degree g, rather than the second factor, which has degree $g+1$, completely into linear factors.) The elements M_j associated to all irreducible factors in $\mathbb{Z}[x]$ are integral by Theorem 8.3, so we get at least $g + 1$ elements of $K_2(C;\mathbb{Z})$, but we have to take (2) of Proposition 6.14 into account.

An equation of the form (10.9) is not unique for a given curve. For even g, we may (and will) assume that the sign '$\pm$' in (10.9) is '$+$', for we can otherwise replace $(x, y, \{v_j\})$ with $(-x, -y, \{-v_{g+1-j}\})$. On the other hand, for odd g the map $(x, y) \mapsto (-x, y)$ gives an isomorphism between the two curves with the same sign associated to $\{v_j\}$ and $\{-v_{g+1-j}\}$.

Also, if we look at the model at infinity by letting $y \mapsto y/x^{g+1}$ and $x \mapsto 1/x$, the equation becomes

$$y^2 + \Big(2 \pm x \prod_{j=1}^{g}(x + v_j)\Big)\, y + 1 = 0\,.$$

Now it is clear that translating x by $-v_j$ for some j gives an equation of the same shape. In other words, $\{v_j\}$ and $\{w_j\}$ yield an isomorphic curve whenever $\{v_j\}\cup\{0\}$

is a translate of $\{w_j\} \cup \{0\}$. Thus, after translating by $-v_1$ in case $v_1 < 0$, we can assume that all v_j are positive. Combining this with the above, we see that for odd g, $0 < v_1 < \cdots < v_g$ and $0 < v_g - v_{g-1} < \cdots < v_g - v_1 < v_g$ give isomorphic curves, and we have chosen the lexicographically smaller representative in Tables 7 and 8 for genus 3 curves.

We can also look at the cases where the last rational factor in (10.10) is reducible, for instance, when it has a linear factor $ax - 1$ with $a \in \mathbb{Z}$. Then $1/a$ is a root of this polynomial, so

$$\prod_{j=1}^{g} (v_j/a + 1) = \mp 4a^{-g-1}.$$

It follows that $a(a+v_1)\cdots(a+v_g) = \mp 4$, which leaves only finitely many possibilities for a and the v_j. It is easy to see that the only two examples for $g > 1$ and $0 < v_1 < v_2 < \ldots$ are

$$v = \{2,5\},\ a = -1 \qquad \text{and} \qquad v = \{3,4\},\ a = -2\,.$$

(There is also $v = \{1,3,4\},\ a = -2$ but the resulting curve is singular.) Finally, there is a case for $g = 3$ where the last factor of (10.10) splits into two quadratic factors over the rationals, namely $v = \{1,5,6\}$. Thus we have found three examples for which the 2-torsion polynomial $t(x)$ has $g+2$ rational factors, all with a '+'-sign in (10.9),

$$\begin{aligned}
v &= \{2,5\}, & -4t(x) &= (x+1)(2x+1)(5x+1)(4x^2+6x+1)\,,\\
v &= \{3,4\}, & -4t(x) &= (2x+1)(3x+1)(4x+1)(2x^2+5x+1)\,,\\
v &= \{1,5,6\}, & -4t(x) &= (x+1)(5x+1)(6x+1)(x^2+6x+1)(4x^2+6x+1)\,.
\end{aligned}$$

In each of the examples, if we associate to these factors elements $M_1, \ldots, M_{g+2}$ of $K_2(C;\mathbb{Z})$ in the order that the factors are written, we can leave out M_{g+2} and still expect a non-trivial relation between the remaining elements. Such a relation was indeed found numerically in all three cases.

The results are summarized in Tables 6 (genus 2), 7 (genus 3), 8 (genus 3) and 9 (genus 4). The entries in the tables are sorted according to the conductor. Unfortunately, we cannot compute the numerical values of $L^*(0)$ for $g \geq 5$ for these curves because the conductors get too large.

Remark 10.11. The second author established in [7] a limit formula for the regulator of $M_1, \ldots, M_g$ (corresponding to $v_1x + 1, \ldots, v_gx + 1$) for the curves in Example 10.8 when $v_1, \ldots, v_{g-1}$ are fixed and $|v_g|$ goes to infinity. It follows that, for every $g \geq 2$, there are infinitely many of such curves for which that regulator does not vanish and $\operatorname{rk} K_2(C;\mathbb{Z}) \geq g$. It is also determined precisely when the curves in Example 10.8 are isomorphic over $\mathbb{Q}$ (see Remark 6.9 of [7]), and shown that the regulator does not vanish (and hence $\operatorname{rk} K_2(C;\mathbb{Z}) \geq g$) for infinitely many isomorphism classes over $\overline{\mathbb{Q}}$.

Such results are also proven for certain hyperelliptic curves over $\mathbb{Q}$ that are similar to the ones in Example 10.8, as well as for certain elliptic curves over arbitrary real quadratic fields.

Remark 10.12. All but two of the curves in our tables have distinct conductors and are therefore pairwise non-isomorphic. The two exceptions are the curve $b = 1$ of Table 2 and the curve $v = \{3,4\}$ of Table 6, which both have conductor 816. These curves are in fact isomorphic, by mapping (x,y) to $(-\frac{x}{2x+1}, \frac{y}{(2x+1)^3})$.

Remark 10.13. As mentioned in the introduction, in a number of cases we gave more than g elements of $K_2(C;\mathbb{Z})$ and hence by Conjecture 3.11 expected to find a linear relation among them. In all but seven cases (two in each of Tables 1 and 6, and one in each of Tables 2, 3 and 7), Proposition 6.3 and (2) of Proposition 6.14 suffice to show that the subgroup Λ of $K_2(C;\mathbb{Z})$ has at most the predicted rank g. Also in these seven cases, the regulator calculations strongly indicate that Λ has rank g, as was worked out in detail for one of the curves in Example 10.1. We did not try to prove the expected relation between the M_i, and therefore that the rank of Λ is as predicted by Conjecture 3.11, for any of the seven cases. Each could, of course, be checked by a finite calculation if it is true: for instance, the conjectural relation (10.3) says that some non-zero multiple of the element

$$\left(\frac{y^2}{x^5}\right) \otimes \left(\frac{(1-x)^{41}(1-4x)^{56}}{(1-6x+11x^2-x^3)^{44}}\right)$$

in the tensor square of the multiplicative group of the function field of the curve $y^2-(7x^2-6x+1)y+x^5=0$ is a combination of tensor products of the form $u_i \otimes (1-u_i)$, and if this is indeed true then one can obviously prove it simply by exhibiting the elements u_i. But in some sense it is precisely the fact that there is no visible reason for such a relation, apart from the fact that the rank is not expected to exceed 2 in this case, that provides the strongest experimental support for Beilinson's conjecture.

Remark 10.14. One reason we have given detailed results for so many individual curves—apart from the fact that each example is non-trivial to find and to calculate and that it therefore seemed worth listing them completely—is that the experimental data about the values of $L^*(C,0)/R(\Lambda)$ may be useful in the future for the formulation or numerical verification of conjectures about this rational number analogous to the Lichtenbaum conjectures in the case of K_n of number fields. We will not make any attempt to give such conjectures. We do note, however, that we often find an integer for $L^*(C,0)/R(\Lambda)$. So it might be that $L^*(C,0)/R(K_2(C;\mathbb{Z}))$ is always an integer, and that the denominators that occur in $L^*(C,0)/R(\Lambda)$ for some of our examples can be explained by the fact that Λ is not the full group $K_2(C;\mathbb{Z})$ because $L^*(C,0)/R(K_2(C;\mathbb{Z})) = (K_2(C;\mathbb{Z}) : \Lambda)\cdot L^*(C,0)/R(\Lambda)$. This would be similar to what happens in the discussion of the case of $b=-1$ in Example 10.1, where $\Lambda_M = \langle M_1, M_2, M_3\rangle$ leads to $L^*(C,0)/R(\Lambda_M) \stackrel{?}{=} 1/2$, but the larger lattice $\Lambda = \langle M_1, M_2, M_3, \mathbb{M}\rangle$, with $(\Lambda : \Lambda_M) = 2$, gives $L^*(C,0)/R(\Lambda) \stackrel{?}{=} 1$.

Unfortunately, we have not been able to test this as we do not know any systematic ways to construct elements of $K_2(C;\mathbb{Z})$ other than the ones we discussed. But it would be interesting to do this, especially for the curve defined by

$$y^2+(5x^3-13x^2+7x-1)y+x^7=0$$

(which is the example with $a=1$ and $b=2$ of Table 3), where the denominator is the relatively large prime 19.

Table 1. Genus 2 curves $y^2 + ((4b-3)x^2 - (5b-1)x + b)y + x^5 = 0$

b	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
–10	$2^2{\cdot}3{\cdot}5^2{\cdot}7{\cdot}11^2{\cdot}1031$	$\langle M_1, M_2\rangle$	161973.28326750	$2^2{\cdot}3{\cdot}5^2{\cdot}7$
–9	$3^3{\cdot}23{\cdot}8461$	$\langle M_1, M_2\rangle$	2647.35530780278	$2^2{\cdot}3^2$
–8	$2^2{\cdot}3{\cdot}61{\cdot}6113$	$\langle M_1, M_2\rangle$	2402.40716016213	$3{\cdot}23/2$
–7	$3{\cdot}7^2{\cdot}53{\cdot}4243$	$\langle M_1, M_2\rangle$	16142.83542365698	$13{\cdot}19$
–6	$2^2{\cdot}3^3{\cdot}5{\cdot}2797$	$\langle M_1, M_2\rangle$	1090.93388314410	$2{\cdot}3^2$
–5	$3{\cdot}5^2{\cdot}37{\cdot}1721$	$\langle M_1, M_2\rangle$	1602.46040140666	29
–4	$2^2{\cdot}3{\cdot}29{\cdot}31^2$	$\langle M_1, M_2\rangle$	196.40102935125	2^2
–3	$3^3{\cdot}7{\cdot}463$	$\langle M_1, M_2\rangle$	41.79313813775	1
–2	$2^2{\cdot}3{\cdot}13{\cdot}173$	$\langle M_1, M_2\rangle$	16.33803025573	1/2
–1	$3{\cdot}5{\cdot}37$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$ $_{41M_1+56M_2-44M_3=0}$	0.22831231665	1
1	$3{\cdot}11^2$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$ $_{4M_1-26M_2+29M_3=0}$	0.12984963472	1/2
2	$2^2{\cdot}3{\cdot}13{\cdot}19$	$\langle M_1, M_2\rangle$	1.90319317513	$1/2^2{\cdot}5$
3	$3^3{\cdot}47$	$\langle M_1, M_2\rangle$	0.62178975664	$1/2{\cdot}3^3$
4	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}223$	$\langle M_1, M_2\rangle$	42.90813731246	1
5	$3{\cdot}5^2{\cdot}43{\cdot}569$	$\langle M_1, M_2\rangle$	802.87262799199	2^4
6	$2^2{\cdot}3^3{\cdot}17^2{\cdot}67$	$\langle M_1, M_2\rangle$	1575.43548439889	$2^2{\cdot}7$
7	$3{\cdot}7^2{\cdot}59{\cdot}1987$	$\langle M_1, M_2\rangle$	6154.43484637856	$2^2{\cdot}5^2$
8	$2^2{\cdot}3{\cdot}67{\cdot}3167$	$\langle M_1, M_2\rangle$	1788.15229247202	3^3
9	$3^3{\cdot}5{\cdot}4733$	$\langle M_1, M_2\rangle$	281.80083335457	2^2
10	$2^2{\cdot}3{\cdot}5^2{\cdot}23{\cdot}83{\cdot}293$	$\langle M_1, M_2\rangle$	85596.822531781	$2^7{\cdot}3^2$

Table 2. Genus 2 curves $y^2 + (2x^3 - 4bx^2 - x + b)y + x^6 = 0$

b	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
1	$2^4{\cdot}3{\cdot}17$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$ $_{10M_1-2M_2+M_3=0}$	0.35283625318	$1/2^3$
2	$2^5{\cdot}3{\cdot}5^2{\cdot}13$	$\langle M_2, M_3\rangle$	22.9767849707	1/2
3	$2^4{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}29$	$\langle M_2, M_3\rangle$	347.644931188	$2{\cdot}3$
4	$2^5{\cdot}3{\cdot}7{\cdot}257$	$\langle M_2, M_3\rangle$	134.428839855	2
5	$2^4{\cdot}3{\cdot}5^2{\cdot}11{\cdot}401$	$\langle M_2, M_3\rangle$	2694.74551646	$2^2{\cdot}3^2$
6	$2^5{\cdot}3^2{\cdot}11{\cdot}13{\cdot}577$	$\langle M_2, M_3\rangle$	20021.3775652	$2{\cdot}3{\cdot}41$
8	$2^5{\cdot}3{\cdot}5^2{\cdot}17{\cdot}41$	$\langle M_2, M_3\rangle$	1106.79592308	$2^2{\cdot}3$
10	$2^5{\cdot}3{\cdot}5^2{\cdot}7{\cdot}19{\cdot}1601$	$\langle M_2, M_3\rangle$	416121.1462	$2^2{\cdot}3{\cdot}7^3$
12	$2^5{\cdot}3^2{\cdot}5^2{\cdot}23{\cdot}461$	$\langle M_2, M_3\rangle$	58663.7341258	$2^2{\cdot}3^3{\cdot}5$
13	$2^4{\cdot}3{\cdot}5^2{\cdot}13^2{\cdot}541$	$\langle M_2, M_3\rangle$	46380.28556	$2{\cdot}3^2{\cdot}23$
14	$2^5{\cdot}3{\cdot}7^2{\cdot}29{\cdot}3137$	$\langle M_2, M_3\rangle$	322837.4973	$2{\cdot}3{\cdot}467$

TABLE 3. Genus 3 curves $y^2 + ((4b-3)x^3 - (4a+5b-1)x^2 + (5a+b)x - a)y + x^7 = 0$

a, b	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
–1,–6	$3{\cdot}5{\cdot}13{\cdot}62773$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	157.32845991764	$2{\cdot}3$
–1,–5	$2{\cdot}3{\cdot}5{\cdot}34543$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	13.71462235525	1
–1,–4	$3^3{\cdot}7^2{\cdot}73$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	0.99948043865	$1/2{\cdot}3$
–1,–2	$3{\cdot}19{\cdot}1051$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	0.68841617094	$1/2{\cdot}7$
–1, 0	$3{\cdot}5{\cdot}7{\cdot}997$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	1.30028297708	$1/2{\cdot}3$
–1, 1	$2{\cdot}3{\cdot}43{\cdot}599$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	1.85769147359	$1/2$
–1, 2	$3^3{\cdot}17{\cdot}199$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	0.95009772024	$1/2{\cdot}3{\cdot}5$
–1, 3	$2{\cdot}3{\cdot}7^2{\cdot}59{\cdot}599$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	117.81139857650	2^2
1, 0	$3{\cdot}29^2{\cdot}71$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	1.91931648711	$1/3$
1, 2	$3{\cdot}13{\cdot}971$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	0.41650834991	$1/19$
1, 3	$2{\cdot}3{\cdot}5^2{\cdot}229$	$\langle M_1, M_2, M_{3,1}, M_{3,2}, \mathbb{M}\rangle$ $_{36M_1-111M_2-41M_{3,1}+71M_{3,2}=0}$	0.33653518886	1
1, 4	$3^2{\cdot}7877$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	0.84449758754	$2/3{\cdot}5$
1, 5	$2{\cdot}3{\cdot}11{\cdot}44071$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	39.13475965836	2
1, 6	$3{\cdot}5^2{\cdot}19{\cdot}46273$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	631.40978672221	$2^2{\cdot}5$
1, 7	$2{\cdot}3^2{\cdot}479{\cdot}2011$	$\langle M_1, M_2, M_3, \mathbb{M}\rangle$	170.28405530697	2^2

TABLE 4. Genus 4 curves $y^2 + f(x)y + x^{10} = 0$

$f(x)$	Conductor	$L^*(0)$	$L^*(0)/R(\Lambda)$
$2x^5+2x^4+x^3+x^2-3x-1$	$2^{11}{\cdot}5^3{\cdot}19{\cdot}29$	35.85879769	$1/2$
$2x^5+2x^4+2x^3-3x^2-2x+1$	$2^{11}{\cdot}3^2{\cdot}17{\cdot}59$	5.336928011	$1/2$
$2x^5+3x^4-3x^3-2x^2-x-1$	$2^3{\cdot}3^3{\cdot}5{\cdot}19{\cdot}331$	1.865694255	$1/2^2{\cdot}3$
$2x^5+3x^4-x^3-4x^2-3x+1$	$2^3{\cdot}3^4{\cdot}5{\cdot}7{\cdot}20759$	126.4283012	1
$2x^5+3x^4+x^3-3x-1$	$2^4{\cdot}3^3{\cdot}7{\cdot}11{\cdot}4793$	41.29358643	$1/2$
$2x^5+4x^4-3x^3-2x+1$	$2^4{\cdot}5^3{\cdot}7{\cdot}11{\cdot}103$	3.546483598	$1/2^3$
$2x^5+4x^4-x^3-3x^2-3x-1$	$2^{10}{\cdot}3{\cdot}7{\cdot}1051$	6.484247251	$1/2^3$
$2x^5+4x^4-x^3-3x^2-x+1$	$2^{12}{\cdot}3^6{\cdot}13$	11.50901911	$1/2^2$
$2x^5+4x^4+3x^3-5x^2-5x-1$	$2^{12}{\cdot}3{\cdot}5{\cdot}19{\cdot}79$	27.69939565	$1/2$
$2x^5+4x^4+3x^3+3x^2-x-1$	$2^{12}{\cdot}3{\cdot}5^2{\cdot}23{\cdot}43$	89.28895569	1
$2x^5+4x^4+5x^3+2x^2+2x+1$	$2^4{\cdot}3^3{\cdot}7^2{\cdot}379$	2.157167657	$1/2^4$

TABLE 5. Genus 5 curves $y^2 + f(x)y + x^{12} = 0$

$f(x)$	Conductor	$L^*(0)$	$L^*(0)/R(\Lambda)$
$2x^6+2x^5-4x^4-3x^3-2x^2+4x-1$	$2^8{\cdot}3^4{\cdot}5{\cdot}7{\cdot}11{\cdot}19$	0.97906446422637	$1/2^2{\cdot}3^3$
$2x^6+4x^5-5x^4-3x^3+2x^2-x-1$	$2^7{\cdot}3^2{\cdot}5^2{\cdot}107{\cdot}139$	2.86707608488323	$1/2^2{\cdot}3$
$2x^6+6x^5-5x^3-3x^2+x+1$	$2^{16}{\cdot}3^2{\cdot}5{\cdot}13^2$	3.21518014215484	$1/2^2{\cdot}3$
$2x^6+2x^5+2x^4-x^3-3x^2-3x-1$	$2^{16}{\cdot}3{\cdot}5^3{\cdot}31$	4.04748393920751	$1/2^3{\cdot}3$
$2x^6+2x^5+x^3-3x^2-x+1$	$2^{16}{\cdot}3^4{\cdot}5{\cdot}181$	28.41118880946	$1/3$

TABLE 6. Genus 2 curves $y^2 + (2x^3+(v_1x+1)(v_2x+1))y + x^6 = 0$

v	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
2, 5	$2^3\cdot3\cdot5^2$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle_{39M_1-26M_2+M_3=0}$	0.246481356638	$1/2^2$
2, 4	$2^6\cdot11$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.339189596082	$1/2^4\cdot3$
3, 4	$2^4\cdot3\cdot17$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle_{9M_1-10M_2+2M_3=0}$	0.352836253176	$1/2^3$
1, 8	$2\cdot7\cdot59$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.412953201772	$1/2^3\cdot7$
1, 2	$2^3\cdot107$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.372337494391	$1/2^3$
1, 4	$2^3\cdot3\cdot53$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.609341856994	$1/2^4$
8, 9	$2\cdot3\cdot223$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.751762610683	$1/2^2\cdot7$
2, 3	$2^3\cdot3\cdot59$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	0.682403286269	$1/2^2\cdot3$
4, 5	$2^3\cdot5\cdot79$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1.466505046768	$1/2^3$
1, 3	$2^3\cdot3\cdot139$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1.573797007363	$1/2^2$
1, 5	$2^4\cdot5\cdot83$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	3.296439983269	$1/2^2$
2, 6	$2^6\cdot3\cdot37$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	4.318055838622	$1/2^2$
5, 8	$2^2\cdot3\cdot5\cdot127$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	4.980029782458	$1/2\cdot3$
1, 9	$2^5\cdot3\cdot149$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	6.585355387353	$1/2^2$
3, 5	$2^3\cdot3\cdot5\cdot229$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	14.988121207002	1
3, 11	$2^2\cdot3\cdot11\cdot229$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	19.618673668593	$1/2$
4, 8	$2^5\cdot997$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	15.361499842059	$1/2$
4, 6	$2^6\cdot3\cdot197$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	19.481730347124	1
3, 6	$2^3\cdot3^5\cdot23$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	20.448409445891	1
2, 8	$2^6\cdot3\cdot269$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	24.399036028269	1
2, 10	$2^6\cdot5^2\cdot37$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	30.231463268350	1
1, 6	$2^3\cdot3\cdot5\cdot499$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	24.839835330300	$3/2$
6, 9	$2^3\cdot3^5\cdot31$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	33.600131091460	1
3, 7	$2^3\cdot3\cdot7\cdot359$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	25.126558038878	1
7, 8	$2^5\cdot7^2\cdot41$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	28.545427660372	$3/2$
5, 6	$2^3\cdot3\cdot5\cdot733$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	43.322633518831	3
3, 12	$2^4\cdot3^5\cdot23$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	42.043168670629	1
1, 7	$2^3\cdot3\cdot7^2\cdot101$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	59.608745714348	3
2, 7	$2^3\cdot5\cdot7^2\cdot67$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	63.124534380486	3
4, 7	$2^4\cdot3\cdot7^2\cdot67$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	76.831677676717	3
6, 8	$2^6\cdot3\cdot829$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	79.443510393961	3
1, 10	$2^3\cdot3\cdot5\cdot1427$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	88.599127797882	3
3, 8	$2^5\cdot3\cdot5^2\cdot73$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	87.679227492015	3
4, 9	$2^3\cdot3\cdot5\cdot1907$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	105.425224481666	3
6, 7	$2^3\cdot3\cdot7\cdot1373$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	101.061871259886	$2\cdot3$
2, 9	$2^3\cdot3\cdot7\cdot11\cdot191$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	164.650770360786	$2\cdot3$
3, 9	$2^3\cdot3^5\cdot199$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	197.318826602911	$2\cdot3$
5, 7	$2^3\cdot5^2\cdot7\cdot353$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	208.897958121197	3^2
5, 9	$2^4\cdot3\cdot5\cdot2089$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	212.013278144012	$2\cdot3$

$9,10$	$2^3\cdot3\cdot5\cdot5261$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	275.183140339336	$2^2\cdot3$
$8,10$	$2^6\cdot5\cdot2221$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	417.499914032374	13
$1,11$	$2^3\cdot5\cdot11\cdot1619$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	294.450550333465	3^2
$4,12$	$2^5\cdot3\cdot83\cdot103$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	412.225445696365	3^2
$7,9$	$2^3\cdot3\cdot7^2\cdot11\cdot73$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	441.365714448209	$3\cdot5$
$2,11$	$2^3\cdot3\cdot11\cdot6053$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	689.050402379536	$3\cdot7$
$5,12$	$2^3\cdot3\cdot5^2\cdot7^2\cdot61$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	860.242245920864	$2\cdot3^2$
$2,12$	$2^6\cdot3\cdot5\cdot2341$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1267.775571888346	$2^2\cdot3^2$
$7,15$	$2\cdot3\cdot5^2\cdot7^2\cdot313$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1075.913430726619	$2\cdot3^2$
$8,17$	$2\cdot3\cdot17\cdot23321$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1201.189781491657	$2\cdot3^2$
$5,10$	$2^3\cdot5^2\cdot59\cdot263$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1440.017545464245	$2^2\cdot3^2$
$4,10$	$2^6\cdot3\cdot5\cdot17\cdot197$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1562.214360456082	$2^3\cdot5$
$6,10$	$2^6\cdot3\cdot5\cdot3797$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	1888.148590577887	$2^4\cdot3$
$4,13$	$2^3\cdot3\cdot11\cdot13^2\cdot89$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	2048.440414253531	$2\cdot3\cdot7$
$8,16$	$2^6\cdot109\cdot601$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	2029.415672068518	2^5
$4,14$	$2^6\cdot5\cdot7^2\cdot373$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	3298.078842289480	2^6
$6,12$	$2^6\cdot3^5\cdot431$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	3779.698402282094	$2\cdot3\cdot13$
$4,11$	$2^4\cdot7\cdot11\cdot23\cdot239$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	3065.942241509888	$2^3\cdot3^2$
$3,10$	$2^3\cdot3\cdot5\cdot7^2\cdot1307$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	3909.737148728376	$2^2\cdot3^3$
$7,10$	$2^3\cdot3\cdot5^2\cdot7\cdot43\cdot59$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	4653.068964642799	$2\cdot3^2\cdot7$
$5,13$	$2^5\cdot5\cdot13\cdot31\cdot263$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	8592.680376054557	$2^3\cdot3\cdot7$
$7,16$	$2^5\cdot3\cdot7^2\cdot4493$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	11344.672982180578	$2^2\cdot3^2\cdot5$
$5,11$	$2^3\cdot3\cdot5\cdot11\cdot26573$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	16672.060582310110	$2\cdot3^3\cdot7$
$6,14$	$2^6\cdot3\cdot7\cdot41\cdot677$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	20507.523505991035	$2^4\cdot23$
$7,14$	$2^3\cdot7^2\cdot117541$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	20242.092778779142	$2^3\cdot3^2\cdot5$
$3,13$	$2^3\cdot3\cdot5^2\cdot13\cdot6553$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	20094.811949554845	$2\cdot3^2\cdot5^2$
$6,16$	$2^6\cdot3\cdot5\cdot73\cdot773$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	26721.552091088677	$2^4\cdot3^3$
$6,15$	$2^3\cdot3^5\cdot5\cdot59\cdot101$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	28628.778269858471	$2\cdot3^5$
$5,14$	$2^3\cdot3\cdot5\cdot7\cdot95621$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	32866.799798423930	$2\cdot3\cdot101$
$5,15$	$2^3\cdot3\cdot5^2\cdot29\cdot4673$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	41114.005206032957	$2^4\cdot3^2\cdot5$
$8,18$	$2^6\cdot3\cdot5^2\cdot73\cdot353$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	59468.612113790229	$2^2\cdot3\cdot71$
$6,13$	$2^3\cdot3\cdot7^2\cdot13\cdot17\cdot619$	$\langle M_1, M_2, 2\mathbb{M}\rangle$	81849.542063836685	$2\cdot3^3\cdot29$

TABLE 7. Genus 3 curves $y^2 + (2x^4+(v_1x+1)(v_2x+1)(v_3x+1))y + x^8 = 0$

v	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
$2,4,6$	$2^8\cdot 3^3\cdot 13$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1.237888702787	$1/2^7$
$1,3,4$	$2^8\cdot 3^2\cdot 7^2$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1.319632692018	$1/2^6$
$1,2,4$	$2^6\cdot 3^3\cdot 67$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1.326260789773	$1/2^5$
$1,2,3$	$2^6\cdot 3^3\cdot 73$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1.360553792128	$1/2^4$
$1,5,6$	$2^8\cdot 3^3\cdot 5^3$	$\langle M_1, M_2, M_3, M_4, 2\mathbb{M}\rangle$ $_{M_1+M_2+11M_3-12M_4=0}$	9.556528296211	$1/2$
$1,3,5$	$2^6\cdot 3^3\cdot 5\cdot 179$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	15.763316182605	$1/2^3$
$3,4,7$	$2^8\cdot 3^2\cdot 7\cdot 113$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	21.282782316338	$1/2^3$
$1,2,5$	$2^8\cdot 3^3\cdot 5\cdot 53$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	18.574599809025	$1/2^2$
$2,6,8$	$2^{14}\cdot 3^2\cdot 17$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	41.533490724724	$1/2^3$
$2,3,5$	$2^6\cdot 3^2\cdot 5^3\cdot 89$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	72.152711581916	1
$1,2,6$	$2^6\cdot 3^3\cdot 5^3\cdot 43$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	105.836805455409	1
$1,3,6$	$2^6\cdot 3^2\cdot 5\cdot 47\cdot 73$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	98.432671578138	$1/2$
$2,3,6$	$2^6\cdot 3^2\cdot 7^2\cdot 373$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	113.826557165808	1
$1,3,7$	$2^6\cdot 3^2\cdot 7\cdot 2837$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	137.788942178887	1
$1,4,6$	$2^6\cdot 3^2\cdot 5\cdot 5669$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	198.333924314685	$3/2$
$1,2,9$	$2^7\cdot 3^3\cdot 7\cdot 883$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	206.791570180633	1
$3,4,8$	$2^5\cdot 3^3\cdot 5\cdot 7^2\cdot 101$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	219.415523333470	1
$4,5,9$	$2^8\cdot 3^3\cdot 5^3\cdot 29$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	278.222780993318	1
$2,4,10$	$2^{11}\cdot 3^3\cdot 5\cdot 101$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	375.379161057633	1
$1,2,8$	$2^6\cdot 3^3\cdot 7\cdot 29\cdot 103$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	429.330911878658	$5/2$
$1,4,9$	$2^7\cdot 3^2\cdot 5\cdot 9281$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	554.543543287698	2
$1,3,9$	$2^6\cdot 3^2\cdot 137437$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	831.548080326194	2^2
$1,5,8$	$2^7\cdot 3^3\cdot 5\cdot 7\cdot 659$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	798.968120685204	3
$1,4,7$	$2^8\cdot 3^2\cdot 7\cdot 5879$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	930.405544167007	5
$1,2,7$	$2^6\cdot 3^3\cdot 5\cdot 7\cdot 1999$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1105.646745724194	2^3
$1,3,8$	$2^7\cdot 3^3\cdot 5^3\cdot 7\cdot 41$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1395.300068143365	2^3
$1,4,8$	$2^5\cdot 3^3\cdot 7\cdot 21773$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1402.036187242234	$2\cdot 3$
$2,4,8$	$2^{11}\cdot 3^3\cdot 4093$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	3341.033608754311	$2^2\cdot 3$

TABLE 8. Genus 3 curves $y^2+(2x^4-(v_1x+1)(v_2x+1)(v_3x+1))y+x^8=0$

v	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
$1,2,3$	$2^6\cdot 3\cdot 5^3\cdot 11$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	2.288179184223	$1/2^3$
$1,2,4$	$2^6\cdot 3\cdot 5^3\cdot 23$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	5.368903572985	$1/2^3$
$1,2,5$	$2^7\cdot 3\cdot 5\cdot 433$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	8.855164567689	$1/2^3$
$2,4,6$	$2^{12}\cdot 3\cdot 5^3$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	19.749659522297	$1/2^3$
$3,4,8$	$2^5\cdot 3\cdot 5\cdot 7\cdot 19\cdot 31$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	21.559124747863	$1/2^4$
$1,4,5$	$2^{10}\cdot 3\cdot 5\cdot 7\cdot 23$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	22.811436896592	$1/2^2$
$1,3,5$	$2^6\cdot 3\cdot 5\cdot 3797$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	42.967686126549	$1/2$
$2,3,7$	$2^7\cdot 3\cdot 5\cdot 7\cdot 397$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	53.453545184856	$1/2^2$
$1,2,10$	$2^6\cdot 3\cdot 5\cdot 11\cdot 523$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	57.955617125753	$1/2^2$
$2,3,6$	$2^6\cdot 3^2\cdot 5^3\cdot 7\cdot 17$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	114.624730232758	$3/2^2$
$1,2,8$	$2^6\cdot 3\cdot 5^3\cdot 7\cdot 67$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	122.595870053997	$3/2^2$
$1,2,6$	$2^6\cdot 3\cdot 5\cdot 14731$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	125.574116230804	$5/2^2$
$1,5,6$	$2^7\cdot 3\cdot 5^2\cdot 41^2$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	140.322800930811	1
$1,2,9$	$2^6\cdot 3\cdot 5^3\cdot 7\cdot 113$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	196.848432921070	1
$2,3,5$	$2^6\cdot 3^3\cdot 5\cdot 13^3$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	190.825343100486	2
$3,4,9$	$2^7\cdot 3^2\cdot 5\cdot 3847$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	227.021533461314	1
$3,4,7$	$2^{10}\cdot 3^3\cdot 5^3\cdot 7$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	242.380339993035	1
$2,3,8$	$2^6\cdot 3^3\cdot 5\cdot 29\cdot 103$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	279.394212255531	1
$2,3,9$	$2^6\cdot 3^2\cdot 5^3\cdot 7\cdot 61$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	357.850908392946	2
$2,4,8$	$2^{11}\cdot 3\cdot 5^3\cdot 53$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	551.668430135999	2
$3,5,6$	$2^6\cdot 3^2\cdot 5\cdot 21737$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	756.149101085614	$2\cdot 3$
$1,4,7$	$2^7\cdot 3^2\cdot 5^3\cdot 7\cdot 109$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1246.076904950150	$2\cdot 3$
$2,6,8$	$2^{15}\cdot 3^3\cdot 5^3$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1351.447382022975	2^2
$1,4,6$	$2^6\cdot 3^3\cdot 5\cdot 18679$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1854.823840399464	$2^2\cdot 3$
$1,2,7$	$2^6\cdot 3\cdot 5\cdot 7\cdot 47\cdot 569$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	1970.441147103680	$3\cdot 5$
$1,3,9$	$2^6\cdot 3^2\cdot 5^3\cdot 4273$	$\langle M_1, M_2, M_3, 2\mathbb{M}\rangle$	3214.588641563687	$2\cdot 7$

TABLE 9. Genus 4 curves $y^2+(2x^5+(v_1x+1)\cdots(v_4x+1))y+x^{10}=0$

v	Conductor	Λ	$L^*(0)$	$L^*(0)/R(\Lambda)$
$1,2,3,4$	$2^9\cdot 3^2\cdot 17\cdot 113$	$\langle M_1, M_2, M_3, M_4, 2\mathbb{M}\rangle$	2.497987723694	$1/2^5\cdot 5$
$2,3,4,6$	$2^8\cdot 3^2\cdot 80251$	$\langle M_1, M_2, M_3, M_4, 2\mathbb{M}\rangle$	45.006091920102	$1/2^5$
$1,2,3,5$	$2^9\cdot 3^2\cdot 5\cdot 10007$	$\langle M_1, M_2, M_3, M_4, 2\mathbb{M}\rangle$	68.192003860287	$1/2^3$

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